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GENERALIZED FIBONACCI NUMBERS OF THE FORM $11x^2+1$

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Abstract. Let $P \geqslant 3$ be an integer and let (U_n) denote generalized Fibonacci sequence defined by $U_0 = 0, U_1 = 1$ and $U_{n+1} = PU_n - U_{n-1}$ for $n \geqslant 1$. In this study, when P is odd, we solve the equation $U_n = 11x^2 + 1$. We show that only U_1 and U_2 may be of the form $11x^2 + 1$.

1. Introduction

Let P and Q be nonzero integers. Generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) are defined by $U_0(P,Q) = 0, U_1(P,Q) = 1$; $V_0(P,Q) = 2, V_1(P,Q) = P$, and $U_{n+1}(P,Q) = PU_n(P,Q) + QU_{n-1}(P,Q)$, $V_{n+1}(P,Q) = PV_n(P,Q) + QV_{n-1}(P,Q)$ for $n \geqslant 1$. $U_n(P,Q)$ and $V_n(P,Q)$ are called n-th generalized Fibonacci number and n-th generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n}(P,Q) = -(-Q)^{-n}U_n(P,Q) \text{ and } V_{-n}(P,Q) = (-Q)^{-n}V_n(P,Q),$$

respectively. Since

$$U_n(-P,Q) = (-1)^{n-1}U_n(P,Q) \text{ and } V_n(-P,Q) = (-1)^n V_n(P,Q),$$

it will be assumed that $P \geqslant 1$. Moreover, we will assume that $P^2 + 4Q > 0$. For $P = Q = 1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n). For $P = 2$ and $Q = 1$, we have Pell and Pell-Lucas sequences (P_n) and (Q_n). For more information about generalized Fibonacci and Lucas sequences one can consult [**8**].

In [**1**], the authors showed that when $a \neq 0$ and b are integers, the equation $U_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. Moreover, they showed that when $a \neq 0$ and $b \neq \pm 2$, the equation $V_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. In [**4**], Keskin, solved the equations $V_n(P, -1) = wx^2 \pm 1$ for $w = 1, 2, 3, 6$ when P is odd. In [**3**], when P is odd, Karaatlı and Keskin solved the equations $V_n(P, -1) = wx^2 \pm 1$ for $w = 5, 7$. In [**6**], Keskin and Öğüt solved

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the equations $U_n(P,-1) = wx^2 + 1$ for $w = 1, 2, 3, 5, 7, 10$ when P is odd. In this study we solve the equation $U_n(P,-1) = 11x^2 + 1$ for odd value of P. We show that only U_1 and U_2 may be of the form $11x^2+1$. Our main result is Theorem 3.1.

We will use the Jacobi symbol throughout this study. Our method is elementary and used by Cohn, Ribenboim and McDaniel in [**2**] and [**10**], respectively.

2. Preliminaries

From now on, instead of $U_n(P,-1)$ and $V_n(P,-1)$, we write U_n and V_n, respectively. Moreover, we will assume that $P \geqslant 3$.

The following lemmas can be proved by induction.

LEMMA 2.1. *If n is a positive integer, then $U_{2n} \equiv n(-1)^{n+1}P \pmod{P^2}$ and $U_{2n+1} \equiv (-1)^n \pmod{P^2}$.*

LEMMA 2.2. *If n is a positive integer, then $V_{2n} \equiv 2(-1)^n \pmod{P}$ and $V_{2n+1} \equiv 0 \pmod{P}$.*

The following theorems are given in [**4**].

THEOREM 2.1. *Let P be odd. If $V_n = kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

THEOREM 2.2. *Let P be odd. Then the equation $U_n = kx^2 + 1$ has only the solution $n = 1$.*

The following lemma is given in [**5**].

LEMMA 2.3. *$11 \mid V_n$ if and only if $11 \mid P$ and n is odd or $P^2 \equiv 3 \pmod{11}$ and $n = 3t$ for some odd integer t.*

Now we give the following theorem from [**9**], which will be useful for solving the equation $U_n = 11x^2 + 1$.

THEOREM 2.3. *Let P be odd. If $V_n = x^2$ for some integer x, then $n = 1$.*

The following two theorems are given in [**11**].

THEOREM 2.4. *Let $n \in \mathbb{N} \cup \{0\}$, m, $r \in \mathbb{Z}$ and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \pmod{U_m} \tag{2.1}$$

and

$$V_{2mn+r} \equiv V_r \pmod{U_m}. \tag{2.2}$$

THEOREM 2.5. *Let $n \in \mathbb{N} \cup \{0\}$ and m, $r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^n U_r \pmod{V_m} \tag{2.3}$$

and

$$V_{2mn+r} \equiv (-1)^n V_r \pmod{V_m}. \tag{2.4}$$

If $n = 2\cdot 2^k a + r$ with a odd, then we get

$$U_n = U_{2\cdot 2^k a+r} \equiv -U_r \pmod{V_{2^k}} \tag{2.5}$$

and

$$V_n = V_{2\cdot 2^k a+r} \equiv -V_r \pmod{V_{2^k}}. \tag{2.6}$$

by (2.3) and (2.4), respectively.

Since $8 \mid U_3$, when P is odd, we get

$$U_{6q+r} \equiv U_r \pmod 8$$

and

$$V_{6q+r} \equiv V_r \pmod 8 \tag{2.7}$$

by (2.1) and (2.2), respectively.

Moreover, when P is odd, an induction method shows that

$$V_{2^k} \equiv 7 \pmod 8$$

and thus

$$\left(\frac{2}{V_{2^k}}\right) = 1$$

and

$$\left(\frac{-1}{V_{2^k}}\right) = -1 \tag{2.8}$$

for all $k \geqslant 1$.

When P is odd and $P^2 \equiv 1, 4 \pmod{11}$ we get

$$\left(\frac{11}{V_{2^k}}\right) = 1 \tag{2.9}$$

for all $k \geqslant 1$. Moreover, we have

$$\left(\frac{P-1}{V_{2^k}}\right) = \left(\frac{P+1}{V_{2^k}}\right) = 1. \tag{2.10}$$

for $k \geqslant 1$. Now we give some identities concerning generalized Fibonacci and Lucas numbers:

$$\begin{gathered} U_{-n} = -U_n \text{ and } V_{-n} = V_n, \\ U_{2n+1} - 1 = U_n V_{n+1}, \\ U_{2n} = U_n V_n, \\ V_n^2 - (P^2-4)U_n^2 = 4, \\ V_{2n} = V_n^2 - 2 \\ V_{3n} = V_n(V_n^2-3) = V_n(V_{2n}-1). \end{gathered} \tag{2.11}$$

If P is odd, then

$$2 \mid U_n \Leftrightarrow 2 \mid V_n \Longleftrightarrow 3 \mid n. \tag{2.12}$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geqslant 0$, and $d = (m, n)$. Then (see [**7**])

$$(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leqslant b. \end{cases} \tag{2.13}$$

3. Main Theorems

From now on, we will assume that n is a positive integer and P is an odd integer.

LEMMA 3.1. *If $11 \mid P$, then $V_n = 11x^2$ has the solution $n = 1$. If $P^2 \equiv 3 \pmod{11}$, then the equation $V_n = 11x^2$ has no solutions.*

PROOF. Assume that $V_n = 11x^2$ for some integer x. By Lemma 2.3, $11 \mid V_n$ if and only if $11 \mid P$ and n is odd or $P^2 \equiv 3 \pmod{11}$ and $n = 3t$ for some odd integer t. Let $11 \mid P$ and n be odd. Then by Theorem 2.1, we get $n = 1$. Now assume that $P^2 \equiv 3 \pmod{11}$ and $n = 3t$ for some odd integer t. Let $t = 4q \pm 1$. Then $n = 12q \pm 3$ and so

$$V_n = V_{12q\pm3} \equiv V_{\pm3} \equiv V_3 \pmod{U_3}$$

by (2.1). Since $8 \mid U_3$, it follows that

$$11x^2 \equiv V_3 \equiv P(P^2 - 3) \pmod 8.$$

Thus, $11x^2 \equiv -2P \pmod 8$, which implies that $x^2 \equiv -6P \pmod 8$. This is impossible since P is odd. □

LEMMA 3.2. *If $V_n = 11kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

PROOF. Let $V_n = 11kx^2$ for some $k \mid P$ with $k > 1$. Since $11 \mid V_n$, n is odd by Lemma 2.3. Let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Then $V_n \equiv V_1, V_3, V_5 \pmod 8$ by (2.7). Then we get $11kx^2 \equiv P, -2P \pmod 8$. On the other hand, we can write $P = kM$, because $k \mid P$. Then we readily obtain $11kMx^2 \equiv PM, -2PM \pmod 8$ implying that $11Px^2 \equiv PM, -2PM \pmod 8$. This implies that $11x^2 \equiv M, -2M \pmod 8$ since $(8, P) = 1$. Thus, we get $x^2 \equiv 3M, 2M \pmod 8$. Using the fact that M is odd, we have $M \equiv 3 \pmod 8$. Since $11 \mid V_n$, it follows that $11 \mid P$ or $P^2 \equiv 3 \pmod{11}$ by Lemma 2.3. Let $n > 1$. Then $n = 4q \pm 1$ for some $q > 0$ and so $n = 2 \cdot 2^r a \pm 1$ with a odd and $r \geqslant 1$. Thus, $11kx^2 = V_n \equiv -V_1 \pmod{V_{2^r}}$ by (2.6). This shows that $11x^2 \equiv -M \pmod{V_{2^r}}$, which implies that

$$\left(\frac{11}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -\left(\frac{M}{V_{2^r}}\right). \tag{3.1}$$

Now let $r = 1$. If $11 \mid P$ or $P^2 \equiv 3 \pmod{11}$, then it can be seen that $\left(\frac{11}{V_2}\right) = \left(\frac{M}{V_2}\right)$. This is impossible by (3.1). Let $r \geqslant 2$. If $P^2 \equiv 3 \pmod{11}$, then it can be seen that $V_{2^r} \equiv -1 \pmod{11}$ and $V_{2^r} \equiv 2 \pmod M$. If $11 \mid P$, then $V_{2^r} \equiv 2 \pmod{11}$ and $V_{2^r} \equiv 2 \pmod M$. In both cases, it is seen that $\left(\frac{11}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right)$, which is impossible by (3.1). Therefore $n = 1$. □

THEOREM 3.1. *If $U_n = 11x^2 + 1$ for some integer x, then $n = 1$ or $n = 2$.*

PROOF. Assume that $U_n = 11x^2 + 1$ for some integer x. If $11 \mid P$, then by Theorem 2.2, we get $n = 1$. Assume that $11 \nmid P$. Let $n > 2$ be even. Then $11x^2 + 1 \equiv 0 \pmod{P}$ by Lemma 2.1. Thus

$$\left(\frac{11}{P}\right) = \left(\frac{-1}{P}\right),$$

i.e.,

$$\left(\frac{P}{11}\right) = 1. \tag{3.2}$$

Now we divide the proof into four cases.

CASE 3.1. Let $P^2 \equiv 1, 4 \pmod{11}$. Since n is even, $n = 4q + r$ for some $q > 0$ with $r = 0, 2$. Thus $n = 2 \cdot 2^k a + r$ with a odd and $k \geqslant 1$. Then

$$11x^2 = -1 + U_n \equiv -1 - U_r \pmod{V_{2^k}}$$

by (2.5). This shows that

$$11x^2 \equiv -1, -(P+1) \pmod{V_{2^k}},$$

which is impossible since $\left(\frac{11}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, and $\left(\frac{P+1}{V_{2^k}}\right) = 1$ by (2.9), (2.8), and (2.10), respectively.

CASE 3.2. Let $P^2 \equiv 3 \pmod{11}$. Then $11 \mid V_3$ and $P \equiv 5 \pmod{11}$ by (3.2). Since n is even $n = 6q + r$ for some $q > 0$ with $r \in \{0, 2, 4\}$. Therefore

$$U_n = U_{6q+r} \equiv \pm U_r \pmod{V_3}$$

by (2.3). Then

$$U_n = U_{6q+r} \equiv \pm U_0, \pm U_2, \pm U_4 \equiv 0, \pm P \pmod{V_3},$$

which implies that $U_n \equiv 0, \pm 5 \pmod{11}$. But this contradicts the fact that $U_n \equiv 1 \pmod{11}$.

CASE 3.3. Let $P^2 \equiv 5 \pmod{11}$. Then $11 \mid U_5$ and $P \equiv 4 \pmod{11}$ by (3.2). Since n is even $n = 6q + r$ for some $q \geqslant 0$ with $r \in \{0, 2, 4\}$. If $n = 6q$, then

$$11x^2 + 1 = U_n = U_{6q} \equiv U_0 \pmod{U_3}.$$

It follows that $11x^2 \equiv -1 \pmod{8}$ by (2.7), which is impossible. If $n = 6q + 2$, then we can write $n = 12t + 2$ or $n = 12t + 8$ for some $t \geqslant 0$. Let $n = 12t + 2$. Since $16 \mid U_6$, we get $11x^2 + 1 = U_n \equiv U_2 \equiv P \pmod{16}$ by (2.1). A simple calculation shows that $11x^2 + 1 \equiv 1, 4, 12, 13 \pmod{16}$. Thus it can be easily seen that $P \equiv 1, 13 \pmod{16}$. Moreover,

$$11x^2 = -1 + U_n \equiv -1 + U_{12t+2} \equiv -1 + U_2 \pmod{U_3}$$

by (2.1). This shows that

$$11x^2 \equiv P - 1 \pmod{P+1},$$

which implies that

$$11x^2 \equiv -2 \pmod{(P+1)/2}.$$

Then it follows that

$$\Big(\frac{11}{(P+1)/2}\Big)=\Big(\frac{-2}{(P+1)/2}\Big).$$

Therefore

$$\Big(\frac{(P+1)/2}{11}\Big)=\Big(\frac{2}{(P+1)/2}\Big).$$

By using the facts that $(P+1)/2\equiv 1,7 \pmod 8$ and $P\equiv 4 \pmod{11}$ we get

$$-1=\Big(\frac{P+1}{11}\Big)\Big(\frac{2}{11}\Big)=\Big(\frac{(P+1)/2}{11}\Big)=\Big(\frac{2}{(P+1)/2}\Big)=1,$$

a contradiction. Let $n=12t+8$. Then $n=12s-4$ with $s>0$. Since $16\mid U_6$, we get $11x^2+1=U_n\equiv -U_4 \pmod{16}$ by (2.1). By using the fact that $11x^2+1\equiv 1,4,12,13 \pmod{16}$, we get $P\equiv 1,5 \pmod{16}$. Assume that $P\equiv 1 \pmod{16}$. Since $n=12s-4$,

$$11x^2=-1+U_n\equiv -1+U_4 \pmod{U_3}$$

by (2.1). Then we have

$$11x^2\equiv -2 \pmod{(P+1)/2},$$

which implies that

$$\Big(\frac{11}{(P+1)/2}\Big)=\Big(\frac{-2}{(P+1)/2}\Big).$$

From here, we get

$$\Big(\frac{(P+1)/2}{11}\Big)=\Big(\frac{2}{(P+1)/2}\Big).$$

Therefore

$$-1=\Big(\frac{P+1}{11}\Big)\Big(\frac{2}{11}\Big)=\Big(\frac{(P+1)/2}{11}\Big)=\Big(\frac{2}{(P+1)/2}\Big)=1,$$

a contradiction. Now assume that $P\equiv 5 \pmod{16}$. Since n is even, $n=10q+r, r\in\{0,2,4,6,8\}$. Using $11\mid U_5$, we get $11x^2+1=U_n\equiv U_r \pmod{11}$ by (2.1). A simple calculation shows that $r=4$. And so $n=10q+4$. On the other hand, since $n=12s-4$, it is easily seen that$n=60k+44$ for some natural number k. Therefore, n can be written as $n=20q_1+4$ for some natural number q_1. Thus by using (2.3), we get

$$U_n=U_{20q_1+4}\equiv U_4 \pmod{V_5},$$

which implies that

$$11x^2\equiv P^3-2P-1 \pmod{P^4-5P^2+5}$$

since $V_5=P(P^4-5P^2+5)$. This shows that

$$\Big(\frac{11}{P^4-5P^2+5}\Big)=\Big(\frac{P^3-2P-1}{P^4-5P^2+5}\Big)=\Big(\frac{(P^3-2P-1)/2}{P^4-5P^2+5}\Big).$$

By using the facts that $(P^3-2P-1)/2\equiv 1 \pmod 8$, $P^4-5P^2+5\equiv 5 \pmod{11}$, $P^4-5P^2+5\equiv 9 \pmod{16}$, and $-3P^2+P+5\equiv 7 \pmod{16}$, we get

$$1=\Big(\frac{5}{11}\Big)=\Big(\frac{P^4-5P^2+5}{11}\Big)=\Big(\frac{11}{P^4-5P^2+5}\Big)=\Big(\frac{(P^3-2P-1)/2}{P^4-5P^2+5}\Big)$$

$$
\begin{aligned}
&= \Big(\frac{P^4-5P^2+5}{(P^3-2P-1)/2}\Big) = \Big(\frac{-3P^2+P+5}{(P^3-2P-1)/2}\Big) = \Big(\frac{(P^3-2P-1)/2}{-3P^2+P+5}\Big) \\
&= \Big(\frac{P^3-2P-1}{-3P^2+P+5}\Big)\Big(\frac{2}{-3P^2+P+5}\Big) = \Big(\frac{P^3-2P-1}{-3P^2+P+5}\Big) \\
&= \Big(\frac{9(P^3-2P-1)}{-3P^2+P+5}\Big) = \Big(\frac{-2(P+2)}{-3P^2+P+5}\Big) = \Big(\frac{-2}{-3P^2+P+5}\Big)\Big(\frac{P+2}{-3P^2+P+5}\Big) \\
&= -\Big(\frac{P+2}{-3P^2+P+5}\Big) = \Big(\frac{-3P^2+P+5}{P+2}\Big) = \Big(\frac{-9}{P+2}\Big) = \Big(\frac{-1}{P+2}\Big) \\
&= -1,
\end{aligned}
$$

a contradiction. If $n = 6q + 4$, then we can write $n = 12t + 4$ or $n = 12t + 10$ for some nonnegative integer t. Let $n = 12t + 10$. Then $n = 12q_1 - 2$ with $q_1 > 0$. Since $16 \mid U_6$, we get $11x^2 + 1 = U_n \equiv -U_2 \pmod{16}$ by (2.1). Using the fact that $11x^2 + 1 \equiv 1, 4, 12, 13 \pmod{16}$, it is seen that $P \equiv 3, 15 \pmod{16}$. Since $11x^2 \equiv -1 + U_{-2} \pmod{U_3}$ by (2.1), we get

$$11x^2 \equiv -(P+1) \pmod{P^2-1}.$$

Therefore

$$11x^2 \equiv -2 \pmod{P-1},$$

which implies that

$$\Big(\frac{11}{(P-1)/2}\Big) = \Big(\frac{-2}{(P-1)/2}\Big),$$

i.e.,

$$\Big(\frac{(P-1)/2}{11}\Big) = \Big(\frac{2}{(P-1)/2}\Big).$$

By using the fact that $(P-1)/2 \equiv 1, 7 \pmod 8$, we get

$$-1 = -\Big(\frac{3}{11}\Big) = \Big(\frac{P-1}{11}\Big)\Big(\frac{2}{11}\Big) = \Big(\frac{(P-1)/2}{11}\Big) = \Big(\frac{2}{(P-1)/2}\Big) = 1,$$

a contradiction. Let $n = 12t + 4$. Since $16 \mid U_6$, we get $U_n \equiv U_4 \pmod{16}$ by (2.1). This shows that $11x^2 + 1 \equiv P^3 - 2P \pmod{16}$. It can be easily seen that $P \equiv 11, 15 \pmod{16}$ since $11x^2 + 1 \equiv 1, 4, 12, 13 \pmod{16}$. Assume that $P \equiv 15 \pmod{16}$. Then

$$11x^2 = -1 + U_n = -1 + U_{12t+4} \equiv -1 + U_4 \equiv -(P+1) \pmod{U_3},$$

which implies

$$11x^2 \equiv -2 \pmod{(P-1)/2}.$$

It follows that

$$\Big(\frac{11}{(P-1)/2}\Big) = \Big(\frac{-2}{(P-1)/2}\Big)$$

and then

$$\Big(\frac{(P-1)/2}{11}\Big) = \Big(\frac{2}{(P-1)/2}\Big).$$

This is impossible since $\left(\frac{2}{(P-1)/2}\right)=1$ and

$$\left(\frac{(P-1)/2}{11}\right)=\left(\frac{P-1}{11}\right)\left(\frac{2}{11}\right)=-1.$$

Now assume that $P\equiv 11 \pmod{16}$. Since n is even, $n=10q+r$ with $r\in\{0,2,4,6,8\}$. Using $11\mid U_5$, we get $11x^2+1=U_n\equiv U_r \pmod{11}$ by (2.1). A simple calculation shows that $r=4$. Thus $n=10q+4$. Since $n=12t+4$, we get $n=60k+4$ for some natural number k. Therefore by using (2.3), it is seen that

$$U_n=U_{60k+4}\equiv U_4 \pmod{V_5},$$

which implies that

$$11x^2\equiv P^3-2P-1 \pmod{P^4-5P^2+5}$$

since $V_5=P(P^4-5P^2+5)$. This shows that

$$\left(\frac{11}{P^4-5P^2+5}\right)=\left(\frac{P^3-2P-1}{P^4-5P^2+5}\right)=\left(\frac{(P^3-2P-1)/4}{P^4-5P^2+5}\right).$$

By using the facts that $(P^3-2P-1)/4\equiv 3 \pmod 4$, $P^4-5P^2+5\equiv 5 \pmod{11}$, $P^4-5P^2+5\equiv 9 \pmod{16}$, and $-3P^2+P+5\equiv 3 \pmod{16}$, we get

$$\begin{aligned}
1&=\left(\frac{5}{11}\right)=\left(\frac{P^4-5P^2+5}{11}\right)=\left(\frac{11}{P^4-5P^2+5}\right)=\left(\frac{(P^3-2P-1)/4}{P^4-5P^2+5}\right)\\
&=\left(\frac{P^4-5P^2+5}{(P^3-2P-1)/4}\right)=\left(\frac{-3P^2+P+5}{(P^3-2P-1)/4}\right)=-\left(\frac{(P^3-2P-1)/4}{-3P^2+P+5}\right)\\
&=-\left(\frac{P^3-2P-1}{-3P^2+P+5}\right)=-\left(\frac{9(P^3-2P-1)}{-3P^2+P+5}\right)=-\left(\frac{-2(P+2)}{-3P^2+P+5}\right)\\
&=-\left(\frac{-2}{-3P^2+P+5}\right)\left(\frac{P+2}{-3P^2+P+5}\right)=-\left(\frac{P+2}{-3P^2+P+5}\right)\\
&=-\left(\frac{-3P^2+P+5}{P+2}\right)=-\left(\frac{-9}{P+2}\right)=-\left(\frac{-1}{P+2}\right)\\
&=-1,
\end{aligned}$$

a contradiction.

Case 3.4. Case IV. Let $P^2\equiv 9 \pmod{11}$. Then $P\equiv 3 \pmod{11}$ by (3.2). Since n is even, $n=6q+r$ for some $q\geqslant 0$ with $r\in\{0,2,4\}$. If $n=6q$, then

$$11x^2+1=U_n=U_{6q}\equiv U_0 \pmod{U_3}.$$

It follows that $11x^2\equiv -1 \pmod 8$ by (2.7), which is impossible. If $n=6q+2$, then we can write $n=12t+2$ or $n=12t+8$ for some nonnegative integer t. Let $n=12t+8$. Then there exists positive integer q_1 such that $n=12q_1-4$. Therefore by using (2.3), it is seen that

$$U_n=U_{12q_1-4}\equiv \pm U_{-4} \pmod{V_2},$$

which implies that

$$11x^2\equiv -1 \pmod{P^2-2}.$$

Thus

$$\left(\frac{11}{P^2-2}\right)=\left(\frac{-1}{P^2-2}\right)$$

and therefore

$$\left(\frac{P^2-2}{11}\right)=1,$$

which is impossible since $P^2-2\equiv 7 \pmod{11}$. Let $n=12t+2$. Since $16\mid U_6$, we get $11x^2+1=U_n\equiv U_2 \pmod{16}$ by (2.1). A simple calculation shows that $11x^2+1\equiv 1,4,12,13 \pmod{16}$. Since $11x^2+1\equiv P \pmod{16}$, we get $P\equiv 1,13$ (mod 16). Moreover,

$$11x^2=-1+U_n=-1+U_{12t+2}\equiv -1+U_2 \pmod{U_3}$$

by (2.1). This shows that

$$11x^2\equiv P-1 \pmod{P+1},$$

which implies

$$11x^2\equiv -2 \pmod{(P+1)/2}.$$

Then it follows that

$$\left(\frac{11}{(P+1)/2}\right)=\left(\frac{-2}{(P+1)/2}\right).$$

Thus

$$\left(\frac{(P+1)/2}{11}\right)=\left(\frac{2}{(P+1)/2}\right).$$

By using the facts that $(P+1)/2\equiv 1,7 \pmod 8$ and $P+1\equiv 4 \pmod{11}$, we get

$$-1=\left(\frac{P+1}{11}\right)\left(\frac{2}{11}\right)=\left(\frac{(P+1)/2}{11}\right)=\left(\frac{2}{(P+1)/2}\right)=1,$$

a contradiction. If $n=6q+4$, then we can write $n=12t+4$ or $n=12t+10$ for some nonnegative integer t. Let $n=12t+4$. Then

$$11x^2=U_n-1=U_{12t+4}-1\equiv \pm U_4-1 \pmod{V_2},$$

which implies that

$$11x^2\equiv -1 \pmod{P^2-2}.$$

Thus

$$\left(\frac{11}{P^2-2}\right)=\left(\frac{-1}{P^2-2}\right)$$

and then

$$\left(\frac{P^2-2}{11}\right)=1.$$

This is impossible since $P^2-2\equiv 7 \pmod{11}$. Let $n=12t+10$. Then $n=12q_1-2$ with $q_1>0$. Since $16\mid U_6$, we get $11x^2+1=U_n\equiv -U_2 \pmod{16}$ by (2.1). Using the fact that $11x^2+1\equiv 1,4,12,13 \pmod{16}$, it is seen that $P\equiv 3,15$ (mod 16). Since n is even, $n=10q+r$ with $r\in\{0,2,4,6,8\}$. Using $11\mid U_5$, we get $11x^2+1=U_n\equiv U_r \pmod{11}$ by (2.1). A simple calculation shows that $r=6$. Thus $n=10q+6$. Since $n=12t+4$, we get $n=60k-14$ for some natural number

k. Thus n can be written as $n = 20s+6$ for some natural number s. Assume that $P \equiv 15 \pmod{16}$. Then by using (2.3), it is seen that

$$U_n = U_{20s+6} \equiv U_6 \pmod{V_5},$$

which implies that

$$11x^2 \equiv P^5 - 4P^3 + 3P - 1 \pmod{P^4 - 5P^2 + 5}$$

since $V_5 = P(P^4 - 5P^2 + 5)$ and $U_6 = P^5 - 4P^3 + 3P$. This shows that

$$\Big(\frac{11}{P^4 - 5P^2 + 5}\Big) = \Big(\frac{P^5 - 4P^3 + 3P - 1}{P^4 - 5P^2 + 5}\Big) = \Big(\frac{P^3 - 2P - 1}{P^4 - 5P^2 + 5}\Big)$$

By using the facts that $P^4 - 5P^2 + 5 \equiv 8 \pmod{11}$, $P^4 - 5P^2 + 5 \equiv 1 \pmod{16}$, $-3P^2 + P + 5 \equiv 1 \pmod{16}$, and $P^3 - 2P - 1 = 2^r a$ with a odd and $r \geqslant 4$, we get

$$\begin{aligned}
-1 &= \Big(\frac{8}{11}\Big) = \Big(\frac{P^4 - 5P^2 + 5}{11}\Big) = \Big(\frac{11}{P^4 - 5P^2 + 5}\Big) = \Big(\frac{P^3 - 2P - 1}{P^4 - 5P^2 + 5}\Big) \\
&= \Big(\frac{2^r a}{P^4 - 5P^2 + 5}\Big) = \Big(\frac{2}{P^4 - 5P^2 + 5}\Big)^r \Big(\frac{a}{P^4 - 5P^2 + 5}\Big) \\
&= \Big(\frac{a}{P^4 - 5P^2 + 5}\Big) = \Big(\frac{P^4 - 5P^2 + 5}{a}\Big) = \Big(\frac{-3P^2 + P + 5}{a}\Big) \\
&= \Big(\frac{a}{-3P^2 + P + 5}\Big) = \Big(\frac{2}{-3P^2 + P + 5}\Big)^r \Big(\frac{a}{-3P^2 + P + 5}\Big) \\
&= \Big(\frac{2^r a}{-3P^2 + P + 5}\Big) = \Big(\frac{P^3 - 2P - 1}{-3P^2 + P + 5}\Big) = \Big(\frac{9(P^3 - 2P - 1)}{-3P^2 + P + 5}\Big) \\
&= \Big(\frac{-2(P+2)}{-3P^2 + P + 5}\Big) = \Big(\frac{-2}{-3P^2 + P + 5}\Big)\Big(\frac{P+2}{-3P^2 + P + 5}\Big) \\
&= \Big(\frac{P+2}{-3P^2 + P + 5}\Big) = \Big(\frac{-3P^2 + P + 5}{P+2}\Big) = \Big(\frac{-9}{P+2}\Big) = \Big(\frac{-1}{P+2}\Big) \\
&= -1,
\end{aligned}$$

a contradiction. Now assume that $P \equiv 3 \pmod{16}$. Then by using (2.3), we get

$$11x^2 = U_n - 1 = U_{60k-14} - 1 \equiv U_{-14} - 1 \equiv -(U_{14} + 1) \pmod{V_{15}},$$

which implies that

$$11x^2 \equiv -(U_{14} + 1) \pmod{P^8 - 7P^6 + 14P^4 - 8P^2 + 1}$$

since $V_{15} = P(P^4 - 5P^2 + 5)(P^2 - 3)(P^8 - 7P^6 + 14P^4 - 8P^2 + 1)$. Moreover, it can be shown that

$$-(U_{14} + 1) \equiv -(P^5 - 5P^3 + 6P + 1) \pmod{P^8 - 7P^6 + 14P^4 - 8P^2 + 1}. \tag{3.3}$$

Therefore we get

$$11x^2 \equiv -(P^5 - 5P^3 + 6P + 1) \pmod{P^8 - 7P^6 + 14P^4 - 8P^2 + 1}$$

by (3.4) and (3.3). Thus

$$\Big(\frac{11}{P^8 - 7P^6 + 14P^4 - 8P^2 + 1}\Big) = \Big(\frac{-(P^5 - 5P^3 + 6P + 1)}{P^8 - 7P^6 + 14P^4 - 8P^2 + 1}\Big),$$

which implies that

$$\left(\frac{P^8-7P^6+14P^4-8P^2+1}{11}\right)=\left(\frac{P^5-5P^3+6P+1}{P^8-7P^6+14P^4-8P^2+1}\right). \tag{3.4}$$

By using the facts that

$$\begin{aligned} P^8-7P^6+14P^4-8P^2+1 &\equiv 2 \pmod{11},\\ P^8-7P^6+14P^4-8P^2+1 &\equiv 1 \pmod{8},\\ P^4+P^3-2P^2-2P-1 &\equiv 3 \pmod{8},\\ P^3+P^2-3P-1 &\equiv 2 \pmod{8}, \end{aligned}$$

and $P^2-P-1\equiv 5 \pmod{8}$, from (3.4), it is seen that

$$\begin{aligned} -1 &= \left(\frac{2}{11}\right)=\left(\frac{P^8-7P^6+14P^4-8P^2+1}{11}\right)=\left(\frac{P^5-5P^3+6P+1}{P^8-7P^6+14P^4-8P^2+1}\right)\\ &= \left(\frac{P-1}{P^8-7P^6+14P^4-8P^2+1}\right)\left(\frac{P^4+P^3-2P^2-2P-1}{P^8-7P^6+14P^4-8P^2+1}\right)\\ &= \left(\frac{(P-1)/2}{P^8-7P^6+14P^4-8P^2+1}\right)\left(\frac{P^8-7P^6+14P^4-8P^2+1}{P^4+P^3-2P^2-2P-1}\right)\\ &= \left(\frac{P^8-7P^6+14P^4-8P^2+1}{(P-1)/2}\right)\left(\frac{-2(P^3+P^2-3P-1)}{P^4+P^3-2P^2-2P-1}\right)\\ &= \left(\frac{1}{(P-1)/2}\right)\left(\frac{-2(P^3+P^2-3P-1)}{P^4+P^3-2P^2-2P-1}\right)\\ &= \left(\frac{-2}{P^4+P^3-2P^2-2P-1}\right)\left(\frac{P^3+P^2-3P-1}{P^4+P^3-2P^2-2P-1}\right)\\ &= \left(\frac{P^3+P^2-3P-1}{P^4+P^3-2P^2-2P-1}\right)=-\left(\frac{(P^3+P^2-3P-1)/2}{P^4+P^3-2P^2-2P-1}\right)\\ &= -\left(\frac{P^4+P^3-2P^2-2P-1}{(P^3+P^2-3P-1)/2}\right)=-\left(\frac{P^2-P-1}{(P^3+P^2-3P-1)/2}\right)\\ &= -\left(\frac{(P^3+P^2-3P-1)/2}{P^2-P-1}\right)=\left(\frac{P^3+P^2-3P-1}{P^2-P-1}\right)\\ &= \left(\frac{1}{P^2-P-1}\right)\\ &= 1, \end{aligned}$$

a contradiction. Therefore $n=2$. Now assume that $n>3$ and n is odd. Then $n=2m+1$ for some $m>1$. Since $U_n=11x^2+1$, we get $11x^2=U_{2m+1}-1=U_mV_{m+1}$ by (2.11). Let m be odd. Then $(U_m,V_{m+1})=1$ by (2.12) and (2.13). Thus

$$U_m=a^2 \text{ and } V_{m+1}=11b^2 \tag{3.5}$$

or

$$U_m=11a^2 \text{ and } V_{m+1}=b^2 \tag{3.6}$$

for some integers a and b. The identities (3.5) and (3.6) are impossible by Lemma 3.1 and Theorem 2.3, respectively. Let m be even. Then $(U_m,V_{m+1})=P$ by (2.13).

Thus

$$U_m = Pa^2 \text{ and } V_{m+1} = 11Pb^2 \tag{3.7}$$

or

$$U_m = 11Pa^2 \text{ and } V_{m+1} = Pb^2. \tag{3.8}$$

for some integers a and b. The identities (3.7) and (3.8) are impossible by Lemma 3.2 and Theorem 2.1, respectively. Therefore $n \leqslant 3$. If $n = 3$, we get $P^2 - 1 = U_3 = 11x^2 + 1$, which implies that $P^2 \equiv 2 \pmod{11}$. This is impossible. Thus $n = 1$. As a consequence, we get $n = 1$ or $n = 2$. □

References

1. M. A. Alekseyev, S. Tengely, *On integral points on biquadratic curves and near-multiples of squares in Lucas sequences*, J. Integer Seq. **17**(6) (2014), Article ID 14.6.6.
2. J. H. E. Cohn, *Squares in some recurrent sequences*, Pac. J. Math. **41** (1972), 631–646.
3. O. Karaatlı, R. Keskin, *Generalized Lucas Numbers of the form $5kx^2$ and $7kx^2$*, Bull. Korean Math. Soc. **52** (2014) 1467–1480.
4. R. Keskin, *Generalized Fibonacci and Lucas Numbers of the form wx^2 and $wx^2 \mp 1$*, Bull. Korean Math. Soc. **51** (2014) 1041–1054.
5. R. Keskin, M. G. Duman, Ü. Öğüt, *Generalized Lucas Numbers of the form wx^2 and wkx^2*, Hokkaido Math. J., (accepted).
6. R. Keskin, Ü. Öğüt, *Generalized Fibonacci Numbers of the form wx^2+1*, Period. Math. Hung. DOI 10.1007/s10998-016-0133-4.
7. W. L. McDaniel, *The g.c.d. in Lucas sequences and Lehmer number sequences*, Fibonacci Q. **29** (1991), 24–30.
8. P. Ribenboim, *My numbers, my friends. Popular lectures on number theory*, Springer, New York, Inc., 2000, 1–41.
9. P. Ribenboim, W. L. McDaniel, *The square terms in Lucas sequences*, J. Number Theory **58** (1996), 104–123.
10. ———, *On Lucas sequence terms of the form kx^2*, Number Theory: Proceedings of the Turku symposium on Number Theory in memory of Kustaa Inkeri (Turku, 1999), de Gruyter, Berlin, 2001, 293–303.
11. Z. Şiar, R. Keskin, *Some new identities concerning generalized Fibonacci and Lucas numbers*, Hacet. J. Math. Stat. **42**(3) (2013), 211–222.

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