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A New Theorem on Absolute Matrix Summability of Fourier Series

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Abstract. We generalize a main theorem dealing with absolute weighted mean summability of Fourier series to the $|A,p_n|_k$ summability factors of Fourier series under weaker conditions. Also some new and known results are obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [**6**])

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geqslant 1$, if (see [**8**],[**10**])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.
Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geqslant 1). \tag{1.4}$$

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The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [**9**]).
The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geqslant 1$, if (see [**1**])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\mid \bar{N}, p_n \mid_k$ summability is the same as $|C, 1|_k$ (resp. $\mid \bar{N}, p_n \mid$) summability.

2. Known Results

Following theorems are dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

THEOREM 2.1. ([**2**]) *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{2.1}$$

Let (X_n) be a positive monotonic nondecreasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{2.4}$$

then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geqslant 1$.

THEOREM 2.2. ([**5**]) *Let (X_n) be a positive monotonic nondecreasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions* (2.1)-(2.3) *and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geqslant 1$.

REMARK It should be noted that condition (2.5) is reduced to the condition (2.4), when $k = 1$. When $k > 1$, condition (2.5) is weaker than condition (2.4) but the converse is not true (see [**5**] for details).

3. An application of absolute matrix summability to Fourier series

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{3.1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geqslant 1$, if (see [**13**])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{3.2}$$

and it is said to be summable $|A, p_n|_k$, $k \geqslant 1$, if (see [**12**])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty. \tag{3.3}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{3.4}$$

If we take $p_n = 1$ for all n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability. Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability .

For any sequence (λ_n) we write that

$$\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} \quad \text{and} \quad \Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty.$$

Let $f(t)$ be a periodic function with period 2π, and Lebesgue integrable over $(-\pi, \pi)$. Write

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) = \sum_{n=0}^{\infty} C_n(x), \tag{3.5}$$

$$\phi(t) = \tfrac{1}{2}\left\{f(x+t) + f(x-t)\right\}, \quad \text{and} \quad \phi_\alpha(t) = \tfrac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \qquad (\alpha > 0).$$

It is well known that if $\phi(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C, 1)$ mean of the sequence $(nC_n(x))$ (see [**7**]).

Many works have been done dealing with absolute summability factors of Fourier series (see [**3**]-[**5**],[**11**]). Among them, in [**5**], Bor has proved the following theorem dealing with the Fourier series.

Theorem 3.1. *If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, (X_n) is a positive monotonic nondecreasing sequence, the sequences (p_n), (λ_n) satisfy the conditions* (2.1)-(2.3) *and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n(x)|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{3.6}$$

then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

If we take $p_n = 1$ for all values of n, then we obtain a new result dealing with $|C,1|_k$ summability factors of Fourier series.

4. Main Results

The aim of this paper is to generalize Theorem 3.1 for $|A, p_n|_k$ summability factors of Fourier series.
Before stating the main theorem, we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{4.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{4.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{4.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{4.4}$$

Theorem 4.1. *Let $k \geqslant 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{4.5}$$

$$a_{n-1,v} \geqslant a_{nv}, \ \text{for } n \geqslant v + 1, \tag{4.6}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{4.7}$$

$$\hat{a}_{n,v+1} = O(v|\Delta_v(\hat{a}_{nv}|). \tag{4.8}$$

If all the conditions of Theorem 3.1 are satisfied, then the series $\sum C_n(x)\lambda_n$ is summable $|A, p_n|_k$, $k \geqslant 1$.

It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 3.1.
We need the following lemma for the proof of our theorem.

Lemma 4.1. ([**2**]) *Under the conditions of Theorem 2.2 we have that*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{4.9}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{4.10}$$

Proof of Theorem 4.1

Let $(I_n(x))$ denotes the A-transform of the series $\sum_{n=1}^{\infty} C_n(x)\lambda_n$. Then, by (4.3) and (4.4), we have

$$\bar{\Delta} I_n(x) = \sum_{v=1}^{n} \hat{a}_{nv} C_v(x)\lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n(x) &= \sum_{v=1}^{n} \hat{a}_{nv} C_v(x)\lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} rC_r(x) + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} rC_r(x) \\
&= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)(v+1)t_v(x) + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n(x) \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1} \frac{t_v(x)}{v} + a_{nn}\lambda_n t_n(x) \frac{n+1}{n} \\
&= I_{n,1}(x) + I_{n,2}(x) + I_{n,3}(x) + I_{n,4}(x).
\end{aligned}$$

To complete the proof of Theorem 4.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,r}(x) \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{4.11}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,1}(x) \mid^k &\leqslant \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{ \sum_{v=1}^{n-1} \left|\frac{v+1}{v}\right| |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| |t_v(x)| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k |t_v(x)|^k \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v(x)|^k \right\}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v(x)|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|\lambda_v||t_v(x)|^k\frac{p_v}{P_v}\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{p_r}{P_r}\frac{|t_r(x)|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}\frac{|t_v(x)|^k}{X_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Now, using Hölder's inequality we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\,I_{n,2}(x)\,|^k &\leqslant \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\frac{v+1}{v}||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v(x)|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{a}_{nv})||t_v(x)|^k\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{a}_{nv})||t_v(x)|^k\\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)|t_v(x)|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\frac{1}{X_v^{k-1}}|t_v(x)|^k(v|\Delta\lambda_v|)\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{p_r}{P_r}\frac{1}{X_r^{k-1}}|t_r(x)|^k+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}\frac{1}{X_v^{k-1}}|t_v(x)|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

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by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Again, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,3}(x)\mid^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v(x)}{v}\right|^k \\
&\leqslant \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v(x)|}{v}\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}||t_v(x)|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v(x)|^k \times \left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v(x)|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^k|t_v(x)|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|t_v(x)|^k|\lambda_{v+1}|^{k-1}|\lambda_{v+1}| \\
&= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|\lambda_{v+1}||t_v(x)|^k\frac{p_v}{P_v} \\
&= O(1) \quad \text{as} \quad m\to\infty, \qquad \blacksquare
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Finally, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,4}(x)|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k|\lambda_n|^k|t_n(x)|^k = O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n|^{k-1}|\lambda_n||t_n(x)|^k \\
&= O(1)\sum_{n=1}^{m}\frac{1}{X_n^{k-1}}|\lambda_n||t_n(x)|^k\frac{p_n}{P_n} = O(1) \quad \text{as} \quad m\to\infty, \qquad \blacksquare
\end{aligned}$$

by virtue of hypotheses of the Theorem 4.1 and Lemma 4.1. This completes the proof of Theorem 4.1.

If we take $a_{nv}=\frac{p_v}{P_n}$ in Theorem 4.1, then we get Theorem 3.1 and if we take $p_n=1$ for all values of n in Theorem 4.1, then we get a new result dealing with the $|A|_k$ summability method. Also, if we take $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n in Theorem 4.1, then we get a result concerning the $\mid C,1\mid_k$ summability methods.

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