A New Theorem on Absolute Matrix Summability of Fourier Series

Şebnem Yildiz

Abstract. We generalize a main theorem dealing with absolute weighted mean summability of Fourier series to the $|A, p_n|_k$ summability factors of Fourier series under weaker conditions. Also some new and known results are obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. By $u_n^{\alpha}$ and $t_n^{\alpha}$ we denote the $n$th Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$ and $(na_n)$, respectively, that is (see [6])

$$u_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v,$$

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v,$$

where

$$A_n^{\alpha} = \frac{(\alpha + 1)(\alpha + 2) \ldots (\alpha + n)}{n!} = O(n^{\alpha}), \quad A_n^{\alpha} = 0 \quad \text{for} \quad n > 0.$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [8],[10])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^{\alpha} - u_{n-1}^{\alpha}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha}|^k < \infty.$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability. Let $(p_n)$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{n-i} = p_{n-i} = 0, \quad i \geq 1).$$

2010 Mathematics Subject Classification: 26D15; 42A24; 40F05; 40G99.

Key words and phrases: Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality.

Communicated by Gradimir Milovanović.
The sequence-to-sequence transformation

\[ t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \]  

(1.5)

defines the sequence \((t_n)\) of the Riesz mean or simply the \((\tilde{N}, p_n)\) mean of the sequence \((s_n)\) generated by the sequence of coefficients \((p_n)\) (see [9]).

The series \( \sum a_n \) is said to be summable \(|\tilde{N}, p_n|_k\), \( k \geq 1 \), if (see [1])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty.
\]  

(1.6)

In the special case when \( p_n = 1 \) for all values of \( n \) (resp. \( k = 1 \)), \(|\tilde{N}, p_n|_k\) summability is the same as \(|C, 1|_k\) (resp. \(|\tilde{N}, p_n|\)) summability.

2. Known Results

Following theorems are dealing with \(|\tilde{N}, p_n|_k\) summability factors of infinite series.

**Theorem 2.1.** ([2]) Let \((p_n)\) be a sequence of positive numbers such that

\[ P_n = O(np_n) \quad \text{as} \quad n \to \infty. \]  

(2.1)

Let \((X_n)\) be a positive monotonic nondecreasing sequence. If the sequences \((X_n)_m\), \((\lambda_n)_m\) and \((p_n)\) satisfy the conditions

\[ \lambda_m X_m = O(1) \quad \text{as} \quad m \to \infty, \]  

(2.2)

\[ \sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad \text{as} \quad m \to \infty, \]  

(2.3)

\[ \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \]  

(2.4)

then the series \( \sum a_n \lambda_n \) is summable \(|\tilde{N}, p_n|_k\), \( k \geq 1 \).

**Theorem 2.2.** ([5]) Let \((X_n)\) be a positive monotonic nondecreasing sequence. If the sequences \((X_n)_m\), \((\lambda_n)_m\), and \((p_n)\) satisfy the conditions (2.1)-(2.3) and

\[ \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k \frac{1}{X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \]  

(2.5)

then the series \( \sum a_n \lambda_n \) is summable \(|\tilde{N}, p_n|_k\), \( k \geq 1 \).

**Remark** It should be noted that condition (2.5) is reduced to the condition (2.4), when \( k = 1 \). When \( k > 1 \), condition (2.5) is weaker than condition (2.4) but the converse is not true (see [5] for details).
3. An application of absolute matrix summability to Fourier series

Let \( A = (a_{nv}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( As = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...
\]

(3.1)

The series \( \sum a_n \) is said to be summable \( |A|_k, k \geq 1 \), if (see [13])

\[
\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty,
\]

(3.2)

and it is said to be summable \( |A, p_n|_k, k \geq 1 \), if (see [12])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta A_n(s)|^k < \infty.
\]

(3.3)

where

\[
\Delta A_n(s) = A_n(s) - A_{n-1}(s).
\]

If we take \( p_n = 1 \) for all \( n \), \( |A, p_n|_k \) summability is the same as \( |A|_k \) summability.

Also, if we take \( a_{nv} = \frac{p_v}{p_n} \), then \( |A, p_n|_k \) summability is the same as \( |\bar{N}, p_n|_k \) summability.

For any sequence \( (\lambda_n) \) we write that

\[
\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \quad \text{and} \quad \Delta \lambda_n = \lambda_n - \lambda_{n+1}.
\]

A sequence \( (\lambda_n) \) is said to be of bounded variation, denoted by \( (\lambda_n) \in BV \), if

\[
\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty.
\]

Let \( f(t) \) be a periodic function with period \( 2\pi \), and Lebesgue integrable over \((-\pi, \pi)\). Write

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x),
\]

\[
\phi(t) = \frac{1}{2} (f(x + t) + f(x - t)) , \quad \text{and} \quad \phi_n(t) = \frac{1}{n^\alpha} \int_0^t (t-u)^{n-1} \phi(u) \, du, \quad (\alpha > 0).
\]

(3.5)

It is well known that if \( \phi(t) \in BV(0, \pi) \), then \( t_n(x) = O(1) \), where \( t_n(x) \) is the \((C,1)\) mean of the sequence \( (nC_n(x)) \) (see [7]).

Many works have been done dealing with absolute summability factors of Fourier series (see [3]-[5],[11]). Among them, in [5], Bor has proved the following theorem dealing with the Fourier series.
THEOREM 3.1. If \( \phi_1(t) \in BV(0, \pi) \), \((X_n)\) is a positive monotonic nondecreasing sequence, the sequences \((p_n)\), \((\lambda_n)\) satisfy the conditions (2.1)-(2.3) and

\[
\sum_{n=1}^{m} \frac{p_n}{P_n} \left| \frac{t_n(x)}{X_{n-1}} \right|^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

then the series \( \sum C_n(x) \lambda_n \) is summable \(| \tilde{N}, P_n |_k\), \( k \geq 1 \).

If we take \( p_n = 1 \) for all values of \( n \), then we obtain a new result dealing with \(|C, 1|_k\) summability factors of Fourier series.

4. Main Results

The aim of this paper is to generalize Theorem 3.1 for \(|A, p_n|_k\) summability factors of Fourier series.

Before stating the main theorem, we must first introduce some further notations.

Given a normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \tilde{A} = (\tilde{a}_{nv}) \) and \( \hat{A} = (\hat{a}_{nv}) \) as follows:

\[
\tilde{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...
\]

and

\[
\hat{a}_{00} = a_{00}, \quad \hat{a}_{n0} = \hat{a}_{nv} - \tilde{a}_{n-1,v}, \quad n = 1, 2, ...
\]

It may be noted that \( \tilde{A} \) and \( \hat{A} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \tilde{a}_{nv} a_v
\]

and

\[
\Delta A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v.
\]

THEOREM 4.1. Let \( k \geq 1 \) and \( A = (a_{nv}) \) be a positive normal matrix such that

\[
\sigma_{\nu 0} = 1, \quad n = 0, 1, ..., \tag{4.5}
\]

\[
a_{n-1,v} \geq a_{nv}, \quad \text{for} \quad n \geq v + 1, \tag{4.6}
\]

\[
a_{nn} = O\left( \frac{P_n}{P_0} \right), \tag{4.7}
\]

\[
\hat{a}_{n,v+1} = O(v |\Delta_e(\hat{a}_{nv})|). \tag{4.8}
\]

If all the conditions of Theorem 3.1 are satisfied, then the series \( \sum C_n(x) \lambda_n \) is summable \(|A, p_n|_k\), \( k \geq 1 \).

It should be noted that if we take \( a_{nv} = \frac{P_v}{P_0} \), then we get Theorem 3.1.

We need the following lemma for the proof of our theorem.
Lemma 4.1. ([2]) Under the conditions of Theorem 2.2 we have that

\[ nX_n|\Delta \lambda_n| = O(1) \quad \text{as} \quad n \to \infty, \]  
\[ \sum_{n=1}^{\infty} X_n|\Delta \lambda_n| < \infty. \]  

Proof of Theorem 4.1

Let \((I_n(x))\) denotes the A-transform of the series \(\sum_{n=1}^{\infty} C_n(x)\lambda_n\). Then, by (4.3) and (4.4), we have

\[ \bar{\Delta}I_n(x) = \sum_{v=1}^{n} \hat{a}_{nv}C_v(x)\lambda_v. \]

Applying Abel’s transformation to this sum, we get that

\[ \bar{\Delta}I_n(x) = \sum_{v=1}^{n} \hat{a}_{nv}C_v(x)\lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v}{v} \right) \sum_{r=1}^{v} rC_r(x) + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} rC_r(x) \]
\[ = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv}\lambda_v}{v} \right)(v+1)t_v(x) + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n(x) \]
\[ = \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv})\lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta \lambda_v t_v(x) \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1} t_{v+1} \frac{v}{v} + a_{nn}\lambda_nt_n(x) \frac{n+1}{n} \]
\[ = I_{n,1}(x) + I_{n,2}(x) + I_{n,3}(x) + I_{n,4}(x). \]

To complete the proof of Theorem 4.1, by Minkowski’s inequality, it is sufficient to show that

\[ \sum_{n=1}^{\infty} \left( \frac{P_n}{P_n} \right)^{k-1} |I_{n,r}(x)|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \]  

First, by applying Hölder’s inequality with indices \(k\) and \(k'\), where \(k > 1\) and \(\frac{1}{k} + \frac{1}{k'} = 1\), we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |I_{n,1}(x)|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} \left| \frac{v+1}{v} \right| |\Delta_v(\hat{a}_{nv})| |\lambda_v||t_v(x)| \right\}^k \]
\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v(x)|^k \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1} \]
\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \hat{a}_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |t_v(x)|^k \right\} \]
inequality we have that

\[ \begin{align*}
&= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v| |t_v(x)|^k \sum_{n=v+1}^{m+1} |\Delta_v(\tilde{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |\lambda_v| |t_v(x)|^k \frac{P_v}{P_v} \\
&= O(1) \sum_{v=1}^{m-1} \sum_{r=1}^{v} \frac{P_r}{P_v} \frac{|t_r(x)|^k}{X_v^{k-1}} + O(1) |\lambda_m| \sum_{v=1}^{m} \frac{P_v}{P_v} |t_v(x)|^k \\
&= O(1) |\Delta_x| X_v + O(1) |\lambda_m| X_m \\
&= O(1) \text{ as } m \to \infty,
\end{align*} \]

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Now, using Hölder’s inequality we have that

\[ \begin{align*}
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |I_{n,2}(x)|^k &\leq \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \left( \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}| |\Delta_v| |t_v(x)| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \left( \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}| |\Delta_v| |t_v(x)| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} (v|\Delta_v|)^k |\Delta_v(\tilde{a}_{nv})| |t_v(x)|^k \times \left( \sum_{v=1}^{n-1} |\Delta_v(\tilde{a}_{nv})| \right)^{-k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} (v|\Delta_v|)^k |\Delta_v(\tilde{a}_{nv})| |t_v(x)|^k \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=1}^{n-1} (v|\Delta_v|)^k |\Delta_v(\tilde{a}_{nv})| |t_v(x)|^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{P_v}{P_v} \frac{1}{|X_v^{k-1}|} |t_v(x)|^k |v| \Delta_v | \\
&= O(1) \sum_{v=1}^{m} \Delta_v | \Delta_v | \sum_{r=1}^{v} \frac{P_r}{P_v} \frac{1}{|X_r^{k-1}|} |t_r(x)|^k + O(1) m |\Delta_m| \sum_{v=1}^{m} \frac{P_v}{P_v} \frac{1}{|X_v^{k-1}|} |t_v(x)|^k \\
&= O(1) \sum_{v=1}^{m} \Delta_v | \Delta_v | X_v + O(1) m |\Delta_m| X_m \\
&= O(1) \sum_{v=1}^{m} \Delta_v | \Delta_v | + O(1) \sum_{v=1}^{m} X_v |\Delta_v| + O(1) m |\Delta_m| X_m \\
&= O(1) \text{ as } m \to \infty,
\end{align*} \]
by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Again, we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k |I_{n,3}(x)|^k \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k \left| \sum_{v=1}^{n-1} a_{n,v+1} \lambda_{v+1} f_v(x) \right|^k \]

\[ \leq \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k \left\{ \sum_{v=1}^{n-1} |a_{n,v+1}||\lambda_{v+1}| |f_v(x)| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_{v+1}| |f_v(x)| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k |\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^{k-1} \sum_{v=1}^{n-1} |\lambda_{v+1}| \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^k \left| \sum_{n=2}^{m+1} \frac{\hat{a}_{nv}}{\lambda_{v+1}} \right|^k \sum_{n=1}^{m+1} |\Delta_v(\hat{a}_{nv})| \]

\[ = O(1) \sum_{n=1}^{m} \frac{P_n}{p_n} |f_v(x)|^k |\lambda_{v+1}|^{k-1} \]

\[ = O(1) \sum_{n=1}^{m} \frac{1}{\lambda_{v+1}} |\lambda_{v+1}| |f_v(x)|^k \]

\[ = O(1) \quad \text{as} \quad m \to \infty, \]

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.1. Finally, as in \( T_{n,1} \), we have that

\[ \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^k |I_{n,4}(x)|^k = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{p_n} \right)^k a_n^{k} |\lambda_n|^k |f_n(x)|^k = O(1) \sum_{n=1}^{m} \frac{P_n}{p_n} |\lambda_n|^{k-1} |\lambda_n||f_n(x)|^k \]

\[ = O(1) \sum_{n=1}^{m} \frac{1}{\lambda_{v+1}} |\lambda_{v+1}| |f_v(x)|^k \frac{P_n}{p_n} = O(1) \quad \text{as} \quad m \to \infty, \]

by virtue of hypotheses of the Theorem 4.1 and Lemma 4.1. This completes the proof of Theorem 4.1.

If we take \( a_{nv} = \frac{P_v}{p_v} \) in Theorem 4.1, then we get Theorem 3.1 and if we take \( p_n = 1 \) for all values of \( n \) in Theorem 4.1, then we get a new result dealing with the \( |A|_k \) summability method. Also, if we take \( a_{nv} = \frac{P_v}{p_v} \) and \( p_n = 1 \) for all values of \( n \) in Theorem 4.1, then we get a result concerning the \( |C,1|_k \) summability methods.
References


Department of Mathematics
Ahi Evran University
Kırşehir, Turkey
sebmenyildiz@ahievran.edu.tr; sebnem.yildiz82@gmail.com