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WILLMORE SPACELIKE SUBMANIFOLDS IN AN INDEFINITE SPACE FORM $N_q^{n+p}(c)$

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ABSTRACT. Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index q $(1 \leqslant q \leqslant p)$ and of constant curvature c. Denote by $\varphi : M \to N_q^{n+p}(c)$ the n-dimensional spacelike submanifold in $N_q^{n+p}(c)$, $\varphi : M \to N_q^{n+p}(c)$ is called a Willmore spacelike submanifold in $N_q^{n+p}(c)$ if it is a critical submanifold to the Willmore functional $W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv$, where S and H denote the norm square of the second fundamental form and the mean curvature of M and $\rho^2 = S - nH^2$. If $q = p$, in [14], we proved some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$. In this paper, we continue to study this topic and prove some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in an indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$.

1. Introduction

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected indefinite space form of index q $(1 \leqslant q \leqslant p)$ and of constant curvature c. If $c > 0$, $c = 0$ or $c < 0$, it is denoted by $S_q^{n+p}(c)$, $\mathbf{R}_q^{n+p}$ or $H_q^{n+p}(c)$. A submanifold M in $N_q^{n+p}(c)$ is said to be spacelike if the induced metric on M from that of the ambient space is positive definite. Let $\varphi : M \to N_q^{n+p}(c)$ be an n-dimensional spacelike submanifold in $N_q^{n+p}(c)$. If $q = p$ and M is a complete maximal spacelike submanifold in $N_p^{n+p}(c)$, from [6], we know that M is totally geodesic for $c \geqslant 0$, thus the class of all such submanifolds are very small. If $0 \leqslant q < p$, from [1] and [4], we know that if M is a complete minimal submanifold in sphere $S^m(c)$ $m > n$, which is embeded in $S_q^{m+q}(c)$ as a totally geodesic spacelike submanifold such that $m - n + q = p$, then M is a complete maximal spacelike submanifold in $S_q^{n+p}(c)$, thus, we see

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that the class of complete maximal spacelike submanifold in $S_q^{n+p}(c)$ is very large. Therefore, if $0 \leqslant q < p$, the topic of studying spacelike submanifold in $S_q^{n+p}(c)$ is also interesting and important. But as far as we know, the results of this topic are less well established. In [1], Alias and Romero studied compact maximal spacelike submanifold M in $S_q^{n+p}(c)$ and proved that if the Ricci curvature of M satisfying $Ric(M) \geqslant (n-1)c$, then M is totally geodesic. Cheng–Ishikawa [4] also studied compact maximal spacelike submanifold in $S_q^{n+p}(c)$ and obtained some important results in terms of the pinching conditions on scalar curvature, sectional curvature and Ricci curvature, respectively.

Denote by $h_{ij}^{\alpha}, S, \vec{H}$ and H the second fundamental form, the norm square of the second fundamental form, the mean curvature vector and the mean curvature of M and denote by ρ^2 the nonnegative function $\rho^2 = S - nH^2$, we define the Willmore functional (see [2, 8, 11]):

$$W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv,$$

which vanishes if and only if M is a totally umbilical spacelike submanifold. It was shown in [9] that this functional is an invariant under the conformal transformations of a conformal space. The points of M are called the critical points of Willmore functional $W(\varphi)$ if $W'(\varphi) = 0$. If the critical points of $W(\varphi)$ are submanifolds in $N_q^{n+p}(c)$, we call them Willmore spacelike submanifolds. Obviously, we notice that the totally umbilical spacelike submanifold is Willmore spacelike submanifold, but, conversely, it is not true.

Since any minimal submanifold in a unit sphere $S^{n+p}(c)$ is not necessarily Willmore submanifold, due to their backgrounds in mathematics, we know that Willmore submanifolds in a unit sphere have been extensively studied in recent years (see [8] and [13]). In indefinite or Lorentzian geometry, we also see that any maximal spacelike submanifold in $N_q^{n+p}(c)$ $(1 \leqslant q \leqslant p)$ is not necessarily Willmore spacelike submanifold, thus the study of Willmore spacelike submanifold in $N_q^{n+p}(c)$ $(1 \leqslant q \leqslant p)$ is also interesting and important. In [14], if $q = p$, we proved some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$. In this paper, we shall continue to study this topic and prove some integral inequalities of Simons' type and rigidity theorems for n-dimensional Willmore spacelike submanifolds in an indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$.

Denote by K and Q the functions which assign to each point of M the infimum of the sectional curvature and the Ricci curvature at the point, we obtain the following:

THEOREM 1.1. *Let $\varphi : M \to N_q^{n+p}(c)$ be an $n(n \geqslant 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N_q^{n+p}(c)$, $c > 0$ and $1 \leqslant q < p$.*

(1) *If $p - q = 1$, then*

$$\int_M \rho^n \Big\{ n(c - H^2) - \Big(2 - \frac{1}{p}\Big)\rho^2 \Big\} dv \leqslant 0. \tag{1.1}$$

In particular, if

$$\rho^2 \leqslant \frac{n}{2-\frac{1}{p}}(c-H^2),$$

then M is totally umbilical or M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$ and is isometric to the Clifford torus $S^k(\frac{1}{\sqrt{2}}c)\times S^k(\frac{1}{\sqrt{2}}c)$;

(2) *If $p-q>1$, then*

$$\int_M \rho^n\Big\{n(c-H^2)-\frac{3}{2}\rho^2\Big\}dv \leqslant 0. \tag{1.2}$$

In particular, if

$$\rho^2 \leqslant \frac{2n}{3}(c-H^2),$$

then M is totally umbilical or M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface.

Theorem 1.2. *Let $\varphi : M \to N_q^{n+p}(c)$ be an $n(n \geqslant 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$. Then the following integral inequality holds*

$$\int_M \rho^n\Big\{K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2\Big\}dv \leqslant 0. \tag{1.3}$$

In particular, if

$$K \geqslant \frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2,$$

then M is totally umbilical or M is a maximal spacelike submanifold in $N_q^{n+p}(c)$ with parallel second fundamental form.

Theorem 1.3. *Let $\varphi : M \to N_q^{n+p}(c)$ be an $n(n \geqslant 2)$-dimensional compact Willmore spacelike submanifold in the indefinite space form $N_q^{n+p}(c)$ $(1 \leqslant q < p)$. Then the following integral inequality holds*

$$\int_M \rho^n\Big\{Q-(n-2)c-nH^2-\frac{1}{n}\Big(3-\frac{p+q}{(p-q)q}\Big)\rho^2\Big\}dv \leqslant 0. \tag{1.4}$$

In particular, if

$$Q \geqslant (n-2)c+nH^2+\frac{1}{n}\Big(3-\frac{p+q}{(p-q)q}\Big)\rho^2,$$

then M is totally umbilical or M is a maximal spacelike submanifold in $N_q^{n+p}(c)$ with parallel second fundamental form.

2. Preliminaries

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional indefinite space form with index $q(1 \leqslant q \leqslant p)$. Let M be an n-dimensional connected spacelike submanifold immersed in $N_q^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots,$ e_{n+p} in $N_q^{n+p}(c)$ such that at each point of M, $e_1, \ldots, e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+p, \quad 1 \leqslant i, j, k, \ldots \leqslant n, \quad n+1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $N_q^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_A = 1$ for $1 \leqslant A \leqslant n+p-q$ and $\varepsilon_A = -1$ for $n+p-q+1 \leqslant A \leqslant n+p$. Then the structure equations of $N_q^{n+p}(c)$ are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c \varepsilon_A \varepsilon_B (\delta_{AD} \delta_{BC} - \delta_{AC} \delta_{BD}). \tag{2.3}$$

If we restrict these form to M, then $\omega_\alpha = 0$, $n+1 \leqslant \alpha \leqslant n+p$ and

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.4}$$

The second fundamental form II, the mean curvature vector $\vec{H}$ of M are defined by

$$II = \sum_{\alpha,i,j} \varepsilon_\alpha h_{ij}^\alpha \omega_i \omega_j e_\alpha, \quad \vec{H} = \sum_\alpha \varepsilon_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha. \tag{2.5}$$

The norm square of the second fundamental form and the mean curvature of M are defined by

$$S = |II|^2 = \sum_{i,j,\alpha} (\varepsilon_\alpha h_{ij}^\alpha)^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad H = |\vec{H}| = \frac{1}{n} \sqrt{\sum_\alpha \Big(\sum_k h_{kk}^\alpha \Big)^2}. \tag{2.6}$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) + \sum_\alpha \varepsilon_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha), \tag{2.7}$$

$$R_{jk} = (n-1) c \delta_{jk} + \sum_\alpha \varepsilon_\alpha \Big(\sum_i h_{ii}^\alpha h_{jk}^\alpha - \sum_i h_{ik}^\alpha h_{ji}^\alpha \Big). \tag{2.8}$$

Defining the first and the second covariant derivatives of h^α_{ij}, say h^α_{ijk} and h^α_{ijkl} by

(2.9)
$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta \varepsilon_\beta h^\beta_{ij}\omega_{\beta\alpha},$$

(2.10)
$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_m h^\alpha_{mjk}\omega_{mi} - \sum_m h^\alpha_{imk}\omega_{mj} - \sum_m h^\alpha_{ijm}\omega_{mk} - \sum_\beta \varepsilon_\beta h^\beta_{ijk}\omega_{\beta\alpha},$$

we have the Codazzi equations and the Ricci identities

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.11}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im}R_{mjkl} - \sum_m h^\alpha_{jm}R_{mikl} - \sum_\beta \varepsilon_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.12}$$

The Ricci equations are

$$R_{\alpha\beta ij} = -\sum_m (h^\alpha_{im}h^\beta_{mj} - h^\alpha_{jm}h^\beta_{mi}). \tag{2.13}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum\limits_k h^\alpha_{ijkk}$. From (2.12), we obtain for any $\alpha, n+1 \leqslant \alpha \leqslant n+p$,

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{k,m} h^\alpha_{km}R_{mijk} - \sum_{k,m} h^\alpha_{im}R_{mkjk} - \sum_{k,\beta} \varepsilon_\beta h^\beta_{ik}R_{\beta\alpha jk}. \tag{2.14}$$

For the fix index $\alpha (n+1 \leqslant \alpha \leqslant n+p)$, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [3] by

$$\Box^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h^\alpha_{ij}) f_{i,j}. \tag{2.15}$$

Since M is compact, the operator $\Box^\alpha$ is self-adjoint (see [3]) if and only if

$$\int_M (\Box^\alpha f) g dv = \int_M f(\Box^\alpha g) dv, \tag{2.16}$$

where f and g are any smooth functions on M. We need the following Lemma (see [12]):

Lemma 2.1. *Let A, B be symmetric $n\times n$ matrices satisfying $AB = BA$ and* $\mathrm{tr}A = \mathrm{tr}B = 0$. *Then*

$$|\mathrm{tr}A^2B| \leqslant \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2}, \tag{2.17}$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy $|x_i| = (\mathrm{tr}B^2)^{1/2}/\sqrt{n(n-1)}$, $x_ix_j \geqslant 0$, $y_i = (\mathrm{tr}A^2)^{1/2}/\sqrt{n(n-1)}$.

By the same method as in the proof of Lemma 4.2 in [8], we also have the following:

LEMMA 2.2. *Let $\varphi : M \to N_q^{n+p}(c)$ be an n-dimensional $(n \geqslant 2)$ spacelike submanifold in $N_q^{n+p}(c)(1 \leqslant q \leqslant p)$. Then we have*

$$|\nabla h|^2 \geqslant \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2, \tag{2.18}$$

where $|\nabla h|^2 = \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha}(H_{,i}^\alpha)^2$.

3. Euler-Lagrange equation of Willmore spacelike submanifolds

From Theorem 4.1 of [9], we know that the Euler-Lagrange equation of Willmore spacelike submanifolds in terms of invariants of conformal metric g is stated as following: a spacelike submanifold is a Willmore spacelike submanifold if and only if

$$\sum_{i,j,k,l,\beta} g_{\alpha\beta}g^{ik}g^{jl}\Big(B_{ij,kl}^\beta + A_{ij}B_{kl}^\beta + \sum_{r,q,\gamma,\nu} g_{\gamma\nu}g^{rq}B_{ir}^\beta B_{qj}^\gamma B_{kl}^\nu\Big) = 0, \quad \forall\alpha, \tag{3.1}$$

where $1 \leqslant i,j,k,l,r,q \leqslant n$, $n+1 \leqslant \alpha,\beta,\gamma,\nu \leqslant n+p$, $(g_{ij}) = (I_n)$, $(g_{\alpha\beta}) = (I_{p-q}) \oplus (-I_q)$, $(g_{ij}) = (g^{ij})^{-1}$ and $(g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$ (see [9]). From (3.23) in [9], we have $(1-n)C_i^\alpha = \sum_k B_{ik,k}^\alpha$. Thus, by a simply calculation, we may rewrite (3.1) as

$$(1-n)\sum_i C_{i,i}^\alpha + \sum_{i,j} A_{ij}B_{ij}^\alpha + \sum_\beta \sum_{i,j,k} \varepsilon_\beta B_{ik}^\alpha B_{kj}^\beta B_{ij}^\beta = 0, \quad \forall\alpha, \tag{3.2}$$

where $\varepsilon_\beta = g_{\beta\beta}$ and $\varepsilon_\beta = 1$ for $n+1 \leqslant \beta \leqslant n+p-q$ and $\varepsilon_\beta = -1$ for $n+p-q+1 \leqslant \beta \leqslant n+p$.

From [9] or [10], we have the following relations of the connections of the conformal metric $e^{2\tau}du \cdot du$ and induced metric $du \cdot du$

$$\omega_i = e^\tau \theta_i, \quad \omega_{ij} = \theta_{ij} + \tau_i\theta_j - \tau_j\theta_i, \quad \omega_{\alpha\beta} = \theta_{\alpha\beta}, \tag{3.3}$$

where $e^{2\tau} = \frac{n}{n-1}(S - nH^2)$. We know that the relations of the conformal invariants and the induced invariants are

$$e^{2\tau}C_i = H^\alpha\tau_i - H_{,i}^\alpha - \sum_j h_{ij}^\alpha \tau^j, \tag{3.4}$$

$$e^{2\tau}A_{ij} = \tau_i\tau_j - \tau_{i,j} - \sum_\alpha H^\alpha h_{ij}^\alpha - \frac{1}{2}\Big(\sum_k \tau^k\tau_k - H^2 - c\Big)I_{ij}, \tag{3.5}$$

$$e^\tau B_{ij}^\alpha = h_{ij}^\alpha - H^\alpha I_{ij}, \tag{3.6}$$

where $\tau_{i,j}$ is Hessian of τ with respect to the first fundamental form I, $\tau^i = \sum_j I^{ij}\tau_j$, $(I^{ij}) = (I_{ij})^{-1}$, $H_i^\alpha = e_i(H^\alpha)$ and $c = 0$ for $\mathbf{R}_q^{n+p}(c)$, $c > 0$ for $S_q^{n+p}(c)$ and $c < 0$ for $H_q^{n+p}(c)$ (see [10])

From (3.3) and (3.4), by a similar calculation of Li [8], we have

$$\sum_j e^\tau C^\alpha_{i,j}\theta_j = \sum_j C^\alpha_{i,j}\omega_j = dC^\alpha_i + \sum_j C^\alpha_j \omega_{ji} + \sum_\beta C^\beta_i \omega_{\beta\alpha}$$
$$= dC^\alpha_i + \sum_j C^\alpha_j \theta_{ji} + \sum_j C^\alpha_j(\tau_j\theta_i - \tau_i\theta_j) + \sum_\beta C^\beta_i \theta_{\beta\alpha},$$

therefore, we have

$$\begin{aligned} e^\tau C^\alpha_{i,j} = e^{-2\tau}\Big(& -2H^\alpha\tau_i\tau_j + 2\tau_j\sum_k h^\alpha_{ik}\tau_k + 2\tau_j H^\alpha_{,i} + H^\alpha_{,j}\tau_i + H^\alpha\tau_{i,j} \\ & - \sum_k h^\alpha_{ik,j}\tau_k - \sum_k h^\alpha_{ik}\tau_{k,j} - H^\alpha_{,ij}\Big) + \sum_k C^\alpha_k\tau_k\delta_{ij} - \tau_i C^\alpha_j. \end{aligned} \tag{3.7}$$

From (3.7), we see that

$$\begin{aligned} e^{3\tau}\sum_i C^\alpha_{i,i} =& (n-3)\Big(H^\alpha|\nabla\tau|^2 - \sum_{i,k} h^\alpha_{ik}\tau_k\tau_i\Big) \\ & - 2(n-2)\sum_i H^\alpha_{,i}\tau_i + H^\alpha\Delta\tau - \sum_{i,k} h^\alpha_{ik}\tau_{k,i} - \Delta^\perp H^\alpha. \end{aligned} \tag{3.8}$$

From (3.5) and (3.6), we have

$$\begin{aligned} & e^{3\tau}\Big(\sum_{i,j} A_{ij}B^\alpha_{ij} + \sum_\beta\sum_{i,j,k}\varepsilon_\beta B^\alpha_{ik}B^\beta_{kj}B^\beta_{ij}\Big) \\ =& \sum_{i,j}\Big[\tau_i\tau_j - \tau_{i,j} - \sum_\beta H^\beta h^\beta_{ij} - \frac{1}{2}\Big(\sum_k \tau^k\tau_k - H^2 - c\Big)I_{ij}\Big]\Big(h^\alpha_{ij} - H^\alpha I_{ij}\Big) \\ & + \sum_\beta\sum_{i,j,k}\varepsilon_\beta\Big(h^\alpha_{ik} - H^\alpha I_{ik}\Big)\Big(h^\beta_{kj} - H^\beta I_{kj}\Big)\Big(h^\beta_{ij} - H^\beta I_{ij}\Big) \\ =& \sum_{i,j} h^\alpha_{ij}\Big(\tau_i\tau_j - \tau_{i,j}\Big) + H^\alpha\Big[\Delta\tau - |\nabla\tau|^2 + n\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2\Big] \\ & + \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij} - H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2. \end{aligned} \tag{3.9}$$

From (3.2), (3.8) and (3.9), we see that

$$\begin{aligned} & (n-2)^2\Big(\sum_{i,j} h^\alpha_{ij}\tau_i\tau_j - H^\alpha|\nabla\tau|^2\Big) + 2(n-1)(n-2)\sum_i H^\alpha_{,i}\tau_i \\ & + (n-2)\Big(\sum_{i,j} h^\alpha_{ij}\tau_{i,j} - H^\alpha\Delta\tau\Big) + (n-1)\Delta^\perp H^\alpha \\ & - H^\alpha\sum_{\beta,i,j}\varepsilon_\beta(h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij}h^\alpha_{ij} \\ & + nH^\alpha\sum_\beta(1+2\varepsilon_\beta)(H^\beta)^2 + \sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij} = 0. \end{aligned} \tag{3.10}$$

Putting $\rho^2 = S - nH^2$, we have $e^{2\tau} = \frac{n}{n-1}(S - nH^2) = \frac{n}{n-1}\rho^2$. Thus $e^\tau = \sqrt{\frac{n}{n-1}}\rho$ and $\tau = \ln(\sqrt{\frac{n}{n-1}}\rho)$. From (3.10), we see that

$$\begin{aligned}
&\frac{\rho^{n-2}}{n-1}\Big\{ - H^\alpha \sum_{\beta,i,j} \varepsilon_\beta (h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij} h^\alpha_{ij} \\
&\quad + \sum_\beta \sum_{i,j,k} \varepsilon_\beta h^\alpha_{ik} h^\beta_{kj} h^\beta_{ij} + nH^\alpha \sum_\beta (1+2\varepsilon_\beta)(H^\beta)^2 \Big\} \\
&\quad + \rho^{n-2}\Delta^\perp H^\alpha + \frac{n-2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_{i,j}\Big(h^\alpha_{ij} - H^\alpha\delta_{ij}\Big) \\
&\quad + 2(n-2)\rho^{n-2}\sum_i (\ln\rho)_i H^\alpha_{,i} \\
&\quad + \frac{(n-2)^2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_i(\ln\rho)_j\Big(h^\alpha_{ij} - H^\alpha\delta_{ij}\Big) = 0.
\end{aligned} \tag{3.11}$$

It can be easily checked that

$$\begin{aligned}
&\rho^{n-2}\Delta^\perp H^\alpha + \frac{n-2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_{i,j}\Big(h^\alpha_{ij} - H^\alpha\delta_{ij}\Big) + 2(n-2)\rho^{n-2}\sum_i(\ln\rho)_i H^\alpha_{,i} \\
&\quad + \frac{(n-2)^2}{n-1}\rho^{n-2}\sum_{i,j}(\ln\rho)_i(\ln\rho)_j\Big(h^\alpha_{ij} - H^\alpha\delta_{ij}\Big) \\
&= -\frac{1}{n-1}\sum_{i,j}(\rho^{n-2})_{i,j}\Big(nH^\alpha\delta_{ij} - h^\alpha_{ij}\Big) + \rho^{n-2}\Delta^\perp H^\alpha \\
&\quad + 2\sum_i(\rho^{n-2})_i H^\alpha_{,i} + H^\alpha\Delta(\rho^{n-2}).
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we may obtain the Euler-Lagrange equation of Willmore spacelike submanifolds in $N^{n+p}_q(c)$ in terms of the induced invariants:

THEOREM 3.1. *Let $\varphi : M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then M is an n-dimensional Willmore spacelike submanifold if and only if for $n+1 \leqslant \alpha, \beta \leqslant n+p$,*

$$\begin{aligned}
&\rho^{n-2}\Big\{ - H^\alpha \sum_{\beta,i,j} \varepsilon_\beta (h^\beta_{ij})^2 - \sum_{\beta,i,j}(1+2\varepsilon_\beta)H^\beta h^\beta_{ij} h^\alpha_{ij} \\
&\quad + \sum_\beta \sum_{i,j,k} \varepsilon_\beta h^\alpha_{ik} h^\beta_{kj} h^\beta_{ij} + nH^\alpha \sum_\beta (1+2\varepsilon_\beta)(H^\beta)^2 \Big\} \\
&\quad + (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i(\rho^{n-2})_i H^\alpha_{,i} \\
&\quad + (n-1)H^\alpha\Delta(\rho^{n-2}) - \Box^\alpha(\rho^{n-2}) = 0.
\end{aligned} \tag{3.13}$$

where $\Delta(\rho^{n-2})=\sum\limits_i(\rho^{n-2})_{i,i}$, $\Box^\alpha(\rho^{n-2})=\sum\limits_{i,j}(\rho^{n-2})_{i,j}(nH^\alpha\delta_{ij}-h^\alpha_{ij})$ *and* $(\rho^{n-2})_{i,j}$ *is the Hessian of* ρ^{n-2} *with respect to the induced metric.*

Remark 3.1. In the proof of (3.13), since we denote $e^{2\tau}=\frac{n}{n-1}(S-nH^2)=\frac{n}{n-1}\rho^2$, it follows that $\rho^2\neq 0$, that is, (3.13) holds only for $\rho^2\neq 0$. But, if $\rho^2=0$, we should notice that (3.13) also holds. Thus, in the following discussion, we agree that the Euler-Lagrange equation of Willmore spacelike submanifolds (3.13) holds for all ρ^2. But, if $n=3$ and $n=5$, we need assume that M has no umbilical points to guarantee $(\rho^{n-2})_{i,j}$ is continuous on M.

Proposition 3.1. *Every maximal spacelike surface* $\varphi: M\to N^{2+p}_q(c)$ *in* $N^{2+p}_q(c)$ *is Willmore spacelike surface.*

In fact, if $n=2$, since $H=0$, from (2.5), we see that $H^\alpha=0$ and $\sum\limits_k h^\alpha_{kk}=0$. On the other hand, since $R_{ij}=\frac{R}{2}\delta_{ij}$, from Gauss equation (2.8), we have $\sum_{\beta,j}\varepsilon_\beta h^\beta_{jk}h^\beta_{ij}=c\delta_{ik}+\sum_{\beta,j}\varepsilon_\beta h^\beta_{jj}h^\beta_{ik}-R_{ik}$, thus

$$\begin{aligned}\sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij}=&\sum_{i,k}h^\alpha_{ik}\Big(c\delta_{ik}+\sum_{\beta,j}\varepsilon_\beta h^\beta_{jj}h^\beta_{ik}-R_{ik}\Big)\\ =&\Big(c-\frac{R}{2}\Big)\sum_i h^\alpha_{ii}+\sum_{\beta,i,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{ik}\Big(\sum_j h^\beta_{jj}\Big)=0,\end{aligned}$$

it follows that (3.13) holds and Proposition 3.1 is concluded.

Example 3.1. If $0\leqslant q<p=q+1$, since we know that the Clifford torus $S^k(\sqrt{\frac{k}{n}}c)\times S^{n-k}(\sqrt{\frac{n-k}{n}}c)$ is a complete minimal hypersurface in sphere $S^{n+1}(c)$ which is embeded in $S^{n+1+q}_q(c)$ as a totally geodesic spacelike submanifold such that $1+q=p$, then $S^k(\sqrt{\frac{k}{n}}c)\times S^{n-k}(\sqrt{\frac{n-k}{n}}c)$ is a complete maximal spacelike submanifold in $S^{n+q+1}_q(c)$, where $1\leqslant k\leqslant n-1$. Since $S^k(\sqrt{\frac{k}{n}}c)\times S^{n-k}(\sqrt{\frac{n-k}{n}}c)$ lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S^{n+q+1}_q(c)$, we know that $h^\alpha_{ij}=0$ for $\alpha=n+2,\dots,n+q+1$. Thus, if and only if $n=2k$ then

$$\begin{aligned}\sum_\beta\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ik}h^\beta_{kj}h^\beta_{ij}=&\sum_{i,j,k}h^{n+1}_{ik}h^{n+1}_{kj}h^{n+1}_{ij}=\sum_i\lambda_i^3\\ =&k\Big(\sqrt{\frac{n-k}{k}c}\Big)^3+(n-k)\Big(-\sqrt{\frac{k}{n-k}c}\Big)^3=0,\end{aligned}$$

where $h^{n+1}_{ij}=\lambda_i\delta_{ij}$, $\sqrt{\frac{n-k}{k}c}$ and $-\sqrt{\frac{k}{n-k}c}$ are the two distinct principal curvatures of $S^k(\sqrt{\frac{k}{n}}c)\times S^{n-k}(\sqrt{\frac{n-k}{n}}c)\subset S^{n+1}(c)$ with multiplicities k and $n-k$, respectively. We also see that $\rho^2=S-nH^2=\sum\limits_i\lambda_i^2=nc$ is constant. Thus, (3.13) holds if and only if $n=2k$, that is the Clifford torus $S^k(\frac{1}{\sqrt{2}}c)\times S^k(\frac{1}{\sqrt{2}}c)$, $1\leqslant k\leqslant n-1$, is a maximal Willmore spacelike submanifold in $S^{n+q+1}_q(c)$.

EXAMPLE 3.2. From [5] and [1], we know that the Veronese surface is a minimal surface in $S^4(c)$ which is embeded in $S_q^{4+q}(c)$ as a totally geodesic spacelike submanifold such that $2+q=p$, then the Veronese surface is a maximal spacelike surface in $S_q^{2+p}(c)$, where $p=2+q$. From Proposition 3.1, we know that it is a Willmore spacelike surface in $S_q^{4+q}(c)$.

4. Basic integral equalities

Define tensors

$$\tilde{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}, \tag{4.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta}, \qquad \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}. \tag{4.2}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{n+1}, \cdots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha}\delta_{\alpha\beta}. \tag{4.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^{\alpha} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^{\alpha}H^{\beta}, \quad \rho^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha} = S - nH^2, \tag{4.4}$$

$$\begin{aligned} &-H^{\alpha}\sum_{\beta,i,j}\varepsilon_{\beta}(h_{ij}^{\beta})^2 - \sum_{\beta,i,j}(1+2\varepsilon_{\beta})H^{\beta}h_{ij}^{\beta}h_{ij}^{\alpha} \\ &+ \sum_{\beta}\sum_{i,j,k}\varepsilon_{\beta}h_{ik}^{\alpha}h_{kj}^{\beta}h_{ij}^{\beta} + nH^{\alpha}\sum_{\beta}(1+2\varepsilon_{\beta})(H^{\beta})^2 \\ &= \sum_{\beta}\sum_{i,j,k}\varepsilon_{\beta}\tilde{h}_{ik}^{\alpha}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\beta} - \sum_{i,j,\beta}H^{\beta}\tilde{h}_{ij}^{\beta}\tilde{h}_{ij}^{\alpha}. \end{aligned} \tag{4.5}$$

From (4.1),(4.4) and (4.5), the Euler-Lagrange equation (3.13) can be rewritten as

PROPOSITION 4.1. *Let $\varphi : M \to N_q^{n+p}(c)$ be an n-dimensional spacelike submanifold in $N_q^{n+p}(c)$. Then M is a Willmore spacelike submanifold if and only if for $n+1 \leqslant \alpha \leqslant n+p$*

(4.6)
$$\begin{aligned} \Box^{\alpha}(\rho^{n-2}) =& (n-1)\rho^{n-2}\Delta^{\perp}H^{\alpha} + 2(n-1)\sum_i(\rho^{n-2})_i H_{,i}^{\alpha} \\ &+ (n-1)H^{\alpha}\Delta(\rho^{n-2}) + \rho^{n-2}\Big(\sum_{\beta}\sum_{i,j,k}\varepsilon_{\beta}\tilde{h}_{ik}^{\alpha}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\beta} - \sum_{i,j,\beta}H^{\beta}\tilde{h}_{ij}^{\beta}\tilde{h}_{ij}^{\alpha}\Big). \end{aligned}$$

Setting $f = nH^{\alpha}$ in (2.15), we have

$$\begin{aligned} \Box^{\alpha}(nH^{\alpha}) &= \sum_{i,j}(nH^{\alpha}\delta_{ij} - h_{ij}^{\alpha})(nH^{\alpha})_{i,j} \\ &= \sum_i (nH^{\alpha})(nH^{\alpha})_{i,i} - \sum_{i,j}h_{ij}^{\alpha}(nH^{\alpha})_{i,j}. \end{aligned} \tag{4.7}$$

We also have

$$\begin{aligned}\frac{1}{2}\Delta(nH)^2 =&\frac{1}{2}\Delta\sum_\alpha(nH^\alpha)^2=\frac{1}{2}\sum_\alpha\Delta(nH^\alpha)^2\\ =&\frac{1}{2}\sum_{\alpha,i}[(nH^\alpha)^2]_{i,i}=\sum_{\alpha,i}[(nH^\alpha)_{,i}]^2+\sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i}\\ =&n^2|\nabla^\perp\vec{H}|^2+\sum_{\alpha,i}(nH^\alpha)(nH^\alpha)_{i,i}.\end{aligned}\tag{4.8}$$

Therefore, from (4.7), (4.8), we get

$$\begin{aligned}\sum_\alpha\Box^\alpha(nH^\alpha)=&\frac{1}{2}\Delta(nH)^2-n^2|\nabla^\perp\vec{H}|^2-\sum_{i,j,\alpha}h_{ij}^\alpha(nH^\alpha)_{i,j}\\ =&\frac{1}{2}\Delta[n(n-1)H^2-\rho^2+S]-n^2|\nabla^\perp\vec{H}|^2-\sum_{i,j,\alpha}h_{ij}^\alpha(nH^\alpha)_{i,j}\\ =&\frac{1}{2}\Delta S+\frac{1}{2}n(n-1)\Delta H^2-\frac{1}{2}\Delta\rho^2-n^2|\nabla^\perp\vec{H}|^2-\sum_{i,j,\alpha}h_{ij}^\alpha(nH^\alpha)_{i,j}.\end{aligned}\tag{4.9}$$

From (2.11) and (2.12), we have

$$\begin{aligned}\frac{1}{2}\Delta S=&\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2+\sum_{i,j,\alpha}h_{ij}^\alpha\Delta h_{ij}^\alpha\\ =&|\nabla h|^2+\sum_{i,j,\alpha}h_{ij}^\alpha(nH^\alpha)_{i,j}-\sum_\alpha\sum_{i,j,k,l}h_{ij}^\alpha h_{kl}^\alpha R_{lijk}\\ &-\sum_\alpha\sum_{i,j,k,l}h_{ij}^\alpha h_{li}^\alpha R_{lkjk}-\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}.\end{aligned}\tag{4.10}$$

Putting (4.10) into (4.9), we have

$$\begin{aligned}\sum_\alpha\Box^\alpha(nH^\alpha)=&|\nabla h|^2-n^2|\nabla^\perp\vec{H}|^2+\frac{1}{2}n(n-1)\Delta H^2-\frac{1}{2}\Delta\rho^2\\ &-\sum_\alpha\sum_{i,j,k,l}h_{ij}^\alpha(h_{kl}^\alpha R_{lijk}+h_{li}^\alpha R_{lkjk})-\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}.\end{aligned}\tag{4.11}$$

Multiplying (4.11) by ρ^{n-2} and taking integration, using (2.16), we have

$$\begin{aligned}\sum_\alpha\int_M(nH^\alpha)\Box^\alpha(\rho^{n-2})dv=&\int_M\rho^{n-2}(|\nabla h|^2-n^2|\nabla^\perp\vec{H}|^2)dv\\ &+\frac{1}{2}n(n-1)\int_M\rho^{n-2}\Delta H^2dv-\frac{1}{2}\int_M\rho^{n-2}\Delta\rho^2dv\\ &-\int_M\rho^{n-2}\sum_\alpha\sum_{i,j,k,l}h_{ij}^\alpha(h_{kl}^\alpha R_{lijk}+h_{li}^\alpha R_{lkjk})dv\end{aligned}\tag{4.12}$$

$$-\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}dv.$$

Taking the Willmore equation (4.6) into (4.12) and making use of the following:

$$\begin{aligned}\int_M \rho^{n-2}\sum_\alpha H^\alpha\triangle^\perp H^\alpha dv =&\frac{1}{2}\int_M \rho^{n-2}\sum_\alpha \Delta^\perp (H^\alpha)^2 dv-\int_M\rho^{n-2}\sum_{i,\alpha}(H^\alpha_{,i})^2dv\\ =&\frac{1}{2}\int_M\rho^{n-2}\Delta H^2dv-\int_M\rho^{n-2}|\nabla\vec{H}|^2dv,\end{aligned}$$

$$\begin{aligned}\int_M H^2\Delta(\rho^{n-2})dv =&\int_M\sum_\alpha (H^\alpha)^2\sum_i(\rho^{n-2})_{i,i}dv\\ =&\sum_{\alpha,i}\int_M (H^\alpha)^2(\rho^{n-2})_{i,i}dv=-\sum_{\alpha,i}\int_M(\rho^{n-2})_i((H^\alpha)^2)_{,i}dv\\ =&-2\int_M\sum_\alpha H^\alpha\sum_i(\rho^{n-2})_iH^\alpha_{,i}dv,\end{aligned}$$

$$\begin{aligned}-\frac{1}{2}\int_M\rho^{n-2}\Delta\rho^2dv=&-\frac{1}{2}\sum_i\int_M\rho^{n-2}(\rho^2)_{i,i}dv\\ =&\frac{1}{2}\sum_i\int_M(\rho^2)_i(\rho^{n-2})_idv=(n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv,\end{aligned}$$

we have, by a direct calculation, the following:

PROPOSITION 4.2. *Let $\varphi: M\to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then*

$$\begin{aligned}\int_M\rho^{n-2}(|\nabla h|^2-n|\nabla^\perp\vec{H}|^2)dv+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv\\ -\int_M\rho^{n-2}\sum_{\alpha,\beta}nH^\alpha\Big(\sum_{i,j,k}\varepsilon_\beta\tilde{h}^\alpha_{ik}\tilde{h}^\beta_{kj}\tilde{h}^\beta_{ij}-H^\beta\tilde{\sigma}_{\alpha\beta}\Big)dv\\ -\int_M\rho^{n-2}\sum_\alpha\sum_{i,j,k,l}h^\alpha_{ij}(h^\alpha_{kl}R_{lijk}+h^\alpha_{li}R_{lkjk})dv\\ -\int_M\rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}dv=0.\end{aligned}\tag{4.13}$$

In general, for a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is, $N(A)=\mathrm{trace}(A\cdot A^t)=\sum_{i,j}(a_{ij})^2$. Clearly, $N(A)=N(T^tAT)$ for any orthogonal matrix T. From (2.13), we have

$$-\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}=-\sum_{\alpha,\beta}\sum_{i,j,k,l}\varepsilon_\beta h^\alpha_{ij}h^\beta_{ki}(h^\beta_{kl}h^\alpha_{lj}-h^\beta_{jl}h^\alpha_{lk})\tag{4.14}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_\beta\Big(\sum_l h^\beta_{kl}h^\alpha_{lj} - \sum_l h^\alpha_{kl}h^\beta_{lj}\Big)^2\\
&= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_\beta\Big(\sum_l \tilde{h}^\beta_{kl}\tilde{h}^\alpha_{lj} - \sum_l \tilde{h}^\alpha_{kl}\tilde{h}^\beta_{lj}\Big)^2\\
&= -\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha),
\end{aligned}
$$

where $\tilde{A}_\alpha := (\tilde{h}^\alpha_{ij}) = (h^\alpha_{ij} - H^\alpha\delta_{ij})$.

By use of (2.7), (2.13), (4.1) (4.2), (4.4) and (4.14), we conclude that

$$
\begin{aligned}
&-\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})\\
&\quad= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\sigma^2_{\alpha\beta} + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta h^\beta_{kj}h^\alpha_{ij}h^\alpha_{ik} - \sum_{\alpha,\beta,i,j,k}\varepsilon_\beta h^\alpha_{ji}h^\beta_{ik}R_{\beta\alpha jk}\\
&\quad= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta} - 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_\beta H^\alpha H^\beta\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij} - n^2\sum_\alpha(H^\alpha)^2\sum_\beta\varepsilon_\beta(H^\beta)^2\\
&\qquad + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} + n\rho^2\sum_\beta\varepsilon_\beta(H^\beta)^2 + 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_\beta H^\alpha H^\beta\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ij}\\
&\qquad + n^2\sum_\alpha(H^\alpha)^2\sum_\beta\varepsilon_\beta(H^\beta)^2 - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha)\\
&\quad= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta} + n\rho^2\sum_\beta\varepsilon_\beta(H^\beta)^2 + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik}\\
&\qquad - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha).
\end{aligned}
\tag{4.15}
$$

Putting (4.14) and (4.15) into (4.13), we have the following:

Proposition 4.3. *Let $\varphi : M \to N^{n+p}_q(c)$ be an n-dimensional spacelike submanifold in $N^{n+p}_q(c)$. Then*

$$
\begin{aligned}
&\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp\vec{H}|^2)dv + (n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv\\
&\quad + n\int_M\rho^{n-2}\Big(\sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} + \rho^2\sum_\beta\varepsilon_\beta(H^\beta)^2\Big)dv + nc\int_M\rho^n dv\\
&\quad - \int_M\rho^{n-2}\Big(\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta}\Big)dv = 0.
\end{aligned}
\tag{4.16}
$$

5. Proofs of Theorems

Proof of Theorem 1.1. (1) If $p-q=1$, from Lemma 2.2 and (4.16), we have

$$
\begin{aligned}
(5.1)\quad 0 = & \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv \\
& + (n-2)\int_M \rho^{n-2}|\nabla \rho|^2 dv \\
& + n\int_M \rho^{n-2}\Big\{\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} + 2\rho^2(H^{n+1})^2 - H^2\rho^2\Big\}dv \\
& + nc\int_M \rho^n dv + \int_M \rho^{n-2}\Big\{\sum_{\alpha=n+2}^{n+p}\sum_{\beta=n+2}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - 2\tilde{\sigma}^2_{n+1n+1} \\
& + \sum_{\alpha=n+1}^{n+p}\sum_{\beta=n+1}^{n+p}\tilde{\sigma}^2_{\alpha\beta}\Big\}dv \\
\geqslant & -n\int_M \rho^{n-2}H^2\rho^2 dv + nc\int_M \rho^n dv - 2\int_M \rho^{n-2}\rho^4 dv + \int_M \rho^{n-2}\frac{1}{p}\rho^4 dv \\
= & \int_M \rho^n\Big\{n(c-H^2) - \Big(2-\frac{1}{p}\Big)\rho^2\Big\}dv,
\end{aligned}
$$

where the inequality $N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \geqslant 0$ for any α, β, $\tilde{\sigma}^2_{n+1n+1} \leqslant \rho^4$ and $\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta} = \sum_\alpha \tilde{\sigma}^2_\alpha \geqslant \frac{1}{p}\Big(\sum_\alpha \tilde{\sigma}_\alpha\Big)^2 = \frac{1}{p}\rho^4$ is used.

In particular, if $\rho^2 \leqslant \frac{n}{2-\frac{1}{p}}(c-H^2)$, from (5.1), we see that $\rho^2 = 0$ and M is totally umbilical or $\rho^2 = \frac{n}{2-\frac{1}{p}}(c-H^2)$. In the latter case, we have from (5.1) that $\int_M \rho^{n-2}\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} dv = 0$, that is

$$
(5.2)\qquad \int_M \rho^{n-2}\sum_\alpha (H^\alpha)^2\tilde{\sigma}_\alpha dv = 0.
$$

If $\rho^2 = 0$, that is M is totally umbilical, otherwise, if $\rho^2 \neq 0$, it follows from (5.2) that $\sum_\alpha (H^\alpha)^2\tilde{\sigma}_\alpha = 0$, thus $(H^\alpha)^2\tilde{\sigma}_\alpha = 0$ for all α. Therefore, we see that $\tilde{\sigma}_\alpha = 0$ for all α (contradicts to $\rho^2 \neq 0$), or $H^\alpha = 0$ for all α. Thus, we have $H = 0$, that is, M is a compact maximal spacelike submanifold in $S^{n+p}_q(c)$, by the Theorem 1 in Cheng and Ishikawa [4] and Example 3.1, we know that M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S^{n+q+1}_q(c)$ and is isometric to the Clifford torus $S^k(\frac{1}{\sqrt{2}}c) \times S^k(\frac{1}{\sqrt{2}}c)$.

(2) If $p-q > 1$, from Lemma 2.2 and (4.16), we have

$$
\begin{aligned}
(5.3)\quad 0 = & \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv \\
& + (n-2)\int_M \rho^{n-2}|\nabla \rho|^2 dv + n\int_M \rho^{n-2}\Big\{\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+2\rho^2\sum_{\beta=n+1}^{n+p-q}(H^\beta)^2-H^2\rho^2\Big\}dv+nc\int_M\rho^n dv\\
&+\int_M\rho^{n-2}\Big\{-\sum_{\alpha,\beta}N(\tilde A_\alpha\tilde A_\beta-\tilde A_\beta\tilde A_\alpha)-\sum_{\alpha,\beta}\tilde\sigma_{\alpha\beta}^2\\
&+2\sum_\alpha\sum_{\beta=n+p-q+1}^{n+p}N(\tilde A_\alpha\tilde A_\beta-\tilde A_\beta\tilde A_\alpha)+2\sum_\alpha\sum_{\beta=n+p-q+1}^{n+p}\tilde\sigma_{\alpha\beta}^2\Big\}dv\\
\geqslant&-n\int_M\rho^{n-2}H^2\rho^2dv+nc\int_M\rho^n dv+\int_M\rho^{n-2}\Big(-\frac{3}{2}\rho^4\Big)dv\\
=&\int_M\rho^n\Big\{n(c-H^2)-\frac{3}{2}\rho^2\Big\}dv,
\end{aligned}
$$

where the inequality (see Li–Li [7])

$$-\sum_{\alpha,\beta}N(\tilde A_\alpha\tilde A_\beta-\tilde A_\beta\tilde A_\alpha)-\sum_{\alpha,\beta}\tilde\sigma_{\alpha\beta}^2\geqslant-\frac{3}{2}\rho^4,$$

is used.

In particular, if $\rho^2\leqslant\frac{2n}{3}(c-H^2)$, from (5.3), we see that $\rho^2=0$ and M is totally umbilical or $\rho^2=\frac{2n}{3}(c-H^2)$. In the latter case, from (5.3), we also see that (5.2) holds. If $\rho^2=0$, that is M is totally umbilical, otherwise, if $\rho^2\neq0$, it follows from (5.2) that $\sum_\alpha(H^\alpha)^2\tilde\sigma_\alpha=0$. By the same argument as above, we see that $H^\alpha=0$ and $H=0$, that is, M is a compact maximal spacelike submanifold in $S_q^{n+p}(c)$, by the Theorem 1 in Cheng and Ishikawa [4] and Example 3.2, we know that M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. For a fixed $\alpha, n+1\leqslant\alpha\leqslant n+p$, we can take a local orthonormal frame field $\{e_1,\dots,e_n\}$ such that $h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij}$, then $\tilde h_{ij}^\alpha=\mu_i^\alpha\delta_{ij}$ with $\mu_i^\alpha=\lambda_i^\alpha-H^\alpha$, $\sum_i\mu_i^\alpha=0$. Thus

$$
\begin{aligned}
(5.4)\qquad -\sum_{\alpha,i,j,k,l}h_{ij}^\alpha(h_{kl}^\alpha R_{lijk}-h_{li}^\alpha R_{lkjk})=&\frac{1}{2}\sum_{\alpha,i,k}(\lambda_i^\alpha-\lambda_k^\alpha)^2R_{kiik}\\
=&\frac{1}{2}\sum_{\alpha,i,k}(\mu_i^\alpha-\mu_k^\alpha)^2R_{kiik}\geqslant nK\rho^2,
\end{aligned}
$$

where K denotes the infimum of the sectional curvature of M and the equality in (5.4) holds if and only if $R_{kiik}=K$ for any $i\neq k$.

Let $\sum_i(\tilde h_{ii}^\beta)^2=\tau_\beta$. Then $\tau_\beta\leqslant\sum_{i,j}(\tilde h_{ij}^\beta)^2=\tilde\sigma_\beta$. Since $\sum_i\tilde h_{ii}^\beta=0$, $\sum_i\mu_i^\alpha=0$ and $\sum_i(\mu_i^\alpha)^2=\tilde\sigma_\alpha$. We have from Lemma 2.1 that

$$(5.5)\qquad -\sum_{\alpha,\beta,i,j,k}H^\alpha\varepsilon_\beta\tilde h_{ik}^\alpha\tilde h_{kj}^\beta\tilde h_{ij}^\beta=-\sum_{\alpha,i,j,k}\sum_{\beta=n+1}^{n+p-q}H^\alpha\tilde h_{ik}^\alpha\tilde h_{kj}^\beta\tilde h_{ij}^\beta$$

$$
\begin{aligned}
&+\sum_{\alpha,i,j,k}\sum_{\beta=n+p-q+1}^{n+p} H^\alpha \tilde{h}_{ik}^\alpha \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\beta \\
=&-\sum_{\alpha,i}\sum_{\beta=n+1}^{n+p-q} H^\alpha \tilde{h}_{ii}^\alpha (\mu_i^\beta)^2 + \sum_{\alpha,i}\sum_{\beta=n+p-q+1}^{n+p} H^\alpha \tilde{h}_{ii}^\alpha (\mu_i^\beta)^2 \\
\geqslant&-\frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha\sum_{\beta=n+1}^{n+p-q} |H^\alpha|\tilde{\sigma}_\beta\sqrt{\tau_\alpha} \\
&-\frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha\sum_{\beta=n+p-q+1}^{n+p} |H^\alpha|\tilde{\sigma}_\beta\sqrt{\tau_\alpha} \\
=&-\frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha |H^\alpha|\sqrt{\tau_\alpha}\Big(\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_\beta + \sum_{\beta=n+p-q+1}^{n+p}\tilde{\sigma}_\beta\Big) \\
\geqslant&-\frac{n-2}{\sqrt{n(n-1)}}\Big(\sqrt{\sum_\alpha (H^\alpha)^2\sum_\alpha\tilde{\tau}_\alpha}\Big)\rho^2 \geqslant -\frac{n-2}{\sqrt{n(n-1)}}H\rho^3.
\end{aligned}
$$

From Lemma 1 in Chen–Do Carmo–Kobayashi [5], we see that

$$
\begin{aligned}
(5.6)\quad &-\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\
=&-\frac{1}{2}\sum_\alpha\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \frac{1}{2}\sum_\alpha\sum_{\beta=n+p-q+1}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\
=&-\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\
&+\frac{1}{2}\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\
\geqslant&-\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \geqslant -\sum_{\alpha\neq\beta}\tilde{\sigma}_\alpha\tilde{\sigma}_\beta \\
=&-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 + \sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha^2 \geqslant -\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 + \frac{1}{p-q}\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 \\
=&-\Big(1-\frac{1}{p-q}\Big)\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2 \geqslant -\Big(1-\frac{1}{p-q}\Big)\rho^4.
\end{aligned}
$$

Thus, from (4.13), (4.14), (5.4), (5.5), (5.6) and Lemma 2.2, we have

$$
(5.7)\quad 0 \geqslant \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv
$$

$$
\begin{aligned}
&+(n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv-\int_M \rho^{n-2}\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho^3 dv\\
&+\int_M \rho^{n-2}\sum_{\alpha,\beta} nH^\alpha H^\beta\tilde{\sigma}_{\alpha\beta}+\int_M \rho^{n-2}nK\rho^2 dv-\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\\
\geqslant&\int_M \rho^n\left\{nK-\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-\Big(1-\frac{1}{p-q}\Big)\rho^2\right\}dv.
\end{aligned}
$$

In particular, if

$$K\geqslant\frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2,$$

from (5.7), we see that $\rho^2=0$ and M is totally umbilical or $K=\frac{n-2}{\sqrt{n(n-1)}}H\rho+\frac{1}{n}\Big(1-\frac{1}{p-q}\Big)\rho^2$. In the latter case, from (5.7), we know that (5.2) holds. If $\rho^2=0$, that is M is totally umbilical, otherwise, if $\rho^2\neq0$, it follows from (5.2) that $\sum_\alpha(H^\alpha)^2\tilde{\sigma}_\alpha=0$. By the same argument as in the proof of Theorem 1.1, we see that $H^\alpha=0$ and $H=0$. It also follows from (5.7) that $|\nabla h|^2=\frac{3n^2}{n+2}|\nabla^\perp\vec{H}|^2=0$, that is, the second fundamental form of M is parallel. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. From (2.8) and (4.1), we have

$$
\begin{aligned}
R_{kk}=&(n-1)c+(n-2)\sum_\alpha\varepsilon_\alpha H^\alpha\tilde{h}^\alpha_{kk}+(n-1)\sum_{\alpha=n+1}^{n+p-q}(H^\alpha)^2\\
&-(n-1)\sum_{\alpha=n+p-q+1}^{n+p}(H^\alpha)^2-\sum_{i,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2+\sum_{i,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2\\
\leqslant&(n-1)c+(n-2)\sum_\alpha\varepsilon_\alpha H^\alpha\tilde{h}^\alpha_{kk}+(n-1)H^2\\
&-\sum_{i,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2+\sum_{i,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2.
\end{aligned}
$$

Thus

$$nQ\leqslant\sum_k R_{kk}=n(n-1)c+n(n-1)H^2-\sum_{i,k,\alpha=n+1}^{n+p-q}(\tilde{h}^\alpha_{ik})^2+\sum_{i,k,\alpha=n+p-q+1}^{n+p}(\tilde{h}^\alpha_{ik})^2.$$

From (4.2) and (4.3), we have

$$-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha+\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\geqslant nQ-n(n-1)c-n(n-1)H^2. \tag{5.8}$$

From (5.8), we see that

(5.9)

$$
\begin{aligned}
&-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2+\frac{1}{q}\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2\\
&\quad=-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2+\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2+\Big(\frac{1}{q}-1\Big)\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2\\
&\quad\geqslant\Big(-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha+\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha+\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)-\Big(1-\frac{1}{q}\Big)\rho^4\\
&\quad\geqslant\Big(nQ-n(n-1)c-n(n-1)H^2\Big)\rho^2-\Big(1-\frac{1}{q}\Big)\rho^4.
\end{aligned}
$$

By Lemma 1 in Chen–Do Carmo–Kobayashi [5], we also see that

$$
-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\geqslant-2\Big(1-\frac{1}{p-q}\Big)\rho^4. \tag{5.10}
$$

From Lemma 2.2, (4.3), (4.16), (5.9) and (5.10), we have

(5.11)

$$
\begin{aligned}
0=&\int_M\rho^{n-2}(|\nabla h|^2-\frac{3n^2}{n+2}|\nabla^\perp\vec{H}|^2)dv+\int_M\rho^{n-2}(\frac{3n^2}{n+2}-n)|\nabla^\perp\vec{H}|^2dv\\
&+(n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv+n\int_M\rho^{n-2}\Big\{\sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta}\\
&+2\rho^2\sum_{\beta=n+1}^{n+p-q}(H^\beta)^2-H^2\rho^2\Big\}dv+nc\int_M\rho^n dv\\
&+\int_M\rho^{n-2}\Big\{-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha^2+\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha^2\Big\}dv\\
\geqslant&-n\int_M\rho^{n-2}H^2\rho^2dv+nc\int_M\rho^n dv+\int_M\rho^{n-2}\Big\{-2\Big(1-\frac{1}{p-q}\Big)\rho^4\Big\}dv\\
&+\int_M\rho^{n-2}\Big\{-\Big(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\Big)^2+\frac{1}{q}\Big(\sum_{\alpha=n+p-q+1}^{n+p}\tilde{\sigma}_\alpha\Big)^2\Big\}dv\\
\geqslant&-n\int_M\rho^{n-2}H^2\rho^2dv+nc\int_M\rho^n dv+\int_M\rho^{n-2}\Big\{-2\Big(1-\frac{1}{p-q}\Big)\rho^4\Big\}dv\\
&+\int_M\rho^{n-2}\Big\{\Big(nQ-n(n-1)c-n(n-1)H^2\Big)\rho^2-\Big(1-\frac{1}{q}\Big)\rho^4\Big\}dv\\
=&\int_M n\rho^n\Big\{Q-(n-2)c-nH^2-\frac{1}{n}\Big(3-\frac{p+q}{(p-q)q}\Big)\rho^2\Big\}dv.
\end{aligned}
$$

In particular, if

$$Q \geqslant (n-2)c + nH^2 + \frac{1}{n}\Big(3 - \frac{p+q}{(p-q)q}\Big)\rho^2,$$

from (5.11), we see that $\rho^2 = 0$ and M is totally umbilical or $Q = (n-2)c + nH^2 + \frac{1}{n}\Big(3 - \frac{p+q}{(p-q)q}\Big)\rho^2$. In the latter case, from (5.11), we know that (5.2) holds. If $\rho^2 = 0$, that is M is totally umbilical, otherwise, if $\rho^2 \neq 0$, it follows from (5.2) that $\sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha = 0$. By the same argument as in the proof of Theorem 1.1, we see that $H^\alpha = 0$ and $H = 0$. It also follows from (5.11) that $|\nabla h|^2 = \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2 = 0$, that is, the second fundamental form of M is parallel. This completes the proof of Theorem 1.3. □

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